

PRACTICAL ERROR ESTIMATES FOR REYNOLDS' LUBRICATION APPROXIMATION AND ITS HIGHER ORDER CORRECTIONS*

JON WILKENING[†]

Abstract. Reynolds' lubrication approximation is used extensively to study flows between moving machine parts, in narrow channels, and in thin films. The solution of Reynolds' equation may be thought of as the zeroth order term in an expansion of the solution of the Stokes equations in powers of the aspect ratio ε of the domain. In this paper, we show how to compute the terms in this expansion to arbitrary order on a two-dimensional, x -periodic domain and derive rigorous, a priori error bounds for the difference between the exact solution and the truncated expansion solution. Unlike previous studies of this sort, the constants in our error bounds either are independent of the function $h(x)$ describing the geometry or depend on h and its derivatives in an explicit, intuitive way. Specifically, if the expansion is truncated at order $2k$, the error is $O(\varepsilon^{2k+2})$, and h enters into the error bound only through its first and third inverse moments $\int_0^1 h(x)^{-m} dx$, $m = 1, 3$, and via the max norms $\|\frac{1}{\ell!} h^{\ell-1} \partial_x^\ell h\|_\infty$, $1 \leq \ell \leq 2k + 2$. We validate our estimates by comparing with finite element solutions and present numerical evidence that suggests that even when h is real analytic and periodic, the expansion solution forms an asymptotic series rather than a convergent series.

Key words. incompressible flow, lubrication theory, asymptotic expansion, Stokes equations, thin domain, a priori error estimates

AMS subject classifications. 76D08, 35C20, 41A80

DOI. 10.1137/070695447

1. Introduction. Reynolds' lubrication equation [22, 20, 16, 12] is used extensively in engineering applications to study flows between moving machine parts, e.g., in journal bearings or computer disk drives. It is also used in microfluid and biofluid mechanics to model creeping flows through narrow channels and in thin films. Although there is a vast literature (including several textbooks) on viscous flows in thin geometries, the equations are normally derived either directly from physical arguments [16] or using formal asymptotic arguments [12]. This is acceptable in most circumstances as the original equations (Stokes or Navier–Stokes) have also been derived from physical considerations, and by now the lubrication equations have been used frequently enough that one can draw on experience and intuition to determine whether they will work well for a given problem.

On the other hand, as soon as the geometry of interest develops (or approaches) a singularity, or if we wish to compute several terms in the asymptotic expansion of the solution in powers of the aspect ratio ε , we rapidly leave the space of problems for which we can use experience as a guide; thus, it would be helpful to have a rigorous proof of convergence to serve as a guide to identify the features of the geometry that could potentially invalidate the approximation. For example, in [25], Wilkening and Hosoi used lubrication theory to study the optimal wave shapes that an animal such as a gastropod should use as it propagates ripples along its muscular foot to crawl over a thin layer of viscous fluid. In certain limits of this constrained optimization problem, the optimal wave shape develops a kink or cusp in the vicinity of the region closest to the substrate, and there is a competing mechanism controlling the size of

* This work was supported in part by the Director, Office of Science, Advanced Scientific Computing Research, U.S. Department of Energy contract DE-AC02-05CH11231.

[†]Department of Mathematics and Lawrence Berkeley National Laboratory, University of California, Berkeley, CA 94720 (wilken@math.berkeley.edu).

the modeling error (singularity formation versus nearness to the substrate). We found that shape optimization within (zeroth order) lubrication theory drives the geometry out of the realm of applicability of the lubrication model; however, by computing higher order corrections and monitoring the errors (using the results of this paper), we learned that cusp-like singularities are appropriately penalized by the full Stokes equations, yielding nonsingular optimal solutions; see [25] for further details.

1.1. Previous work. In most of the following papers, the Stokes or Navier–Stokes equations are solved in a domain Ω_ε bounded below by a flat substrate and above by a curved boundary $y = \varepsilon h(x)$ in two dimensions, or $z = \varepsilon h(x, y)$ in three dimensions, where ε is a small parameter and the function h is fixed. These solutions are then compared to the solution of Reynolds’ equation (or to a truncated expansion solution of the Stokes or Navier–Stokes equations), and the error is shown to converge to zero in the limit as $\varepsilon \rightarrow 0$.

In 1983, Cimatti [8] used a stream function formulation to compare the solution of Reynolds’ equation to that of the Stokes equation in two dimensions. The key idea of the proof, which all subsequent studies (including this one) also use, is that the Poincaré–Friedrichs inequality holds uniformly as $\varepsilon \rightarrow 0$ for the rescaled biharmonic equation (where the domain $\Omega = \Omega_{\varepsilon=1}$ is held fixed and the equations contain the small parameter). Cimatti assumes h has four weak derivatives (whereas, we require only $h \in C^{1,1}$) and shows that for any compact set $K \subset \Omega$,

$$(1.1) \quad \|\varepsilon u - \bar{u}\|_{L^2(\Omega)} \leq C\varepsilon, \quad \max(\|\varepsilon^3 p_x - \bar{p}_x\|_{L^2(K)}, \|\varepsilon^2 p_y\|_{L^2(K)}) \leq C\varepsilon^{1/2},$$

where u is the x -component of velocity, p is the pressure, a bar denotes the solution of Reynolds’ equation, and C is independent of ε but depends on h in the first inequality and on h and K in the second. The scaling here is not standard: he imposes the boundary condition $\varepsilon u(x, 0) = \bar{u}(x, 0) = \text{const}$, which accounts for the extra factor of ε in each of the left-hand sides of (1.1). There are a few problems with Cimatti’s analysis, notably the dependence of C on L (the “arbitrary cutoff” used to make the unbounded domain bounded) and the fact that some of his arguments seem to require ε to be small in comparison to C^{-1} ; however, his basic approach is interesting and inspired much of the work that followed in this subject.

In 1986, Bayada and Chambat [3] generalized Cimatti’s work to three dimensions. They analyze the Stokes equations directly rather than using a stream function formulation, assume less regularity of h (apparently only $h \in C^1$), and state their results in terms of limits (i.e., the quantities u_i^ε , $\varepsilon \partial_x u_i^\varepsilon$, $\partial_y u_i^\varepsilon$, and p^ε in the solution of the Stokes equations converge in L^2 to the corresponding quantities in the solution of Reynolds equations as $\varepsilon \rightarrow 0$); hence, they do not give rates of convergence. In a later paper [4], they also studied the asymptotics of the solution at a junction between a three-dimensional Stokes flow and a thin film flow.

In 1990, Nazarov [18] generalized previous work to the case of the Navier–Stokes equations and also showed how to treat higher order corrections in an asymptotic expansion in the small parameter ε . He proved that if $h(x, y)$ is smooth, then there is a constant C depending on h , N , and the boundary conditions such that

$$(1.2) \quad \|\mathbf{u} - \mathbf{u}^N\|_{H^1} + \|p - \varepsilon^{-1} p^N\|_{L^2} \leq C\varepsilon^{N-1/2},$$

where (\mathbf{u}, p) is the solution of the Navier–Stokes equations, \mathbf{u}^N and p^N are the terms of the asymptotic expansion truncated at the N th order (including a boundary layer expansion near the lateral edges of the thin domain), and the norms are taken on the

thin domain Ω_ε (rather than the rescaled domain Ω). As a corollary, if the expansion is computed with “superfluous” terms that are afterwards treated as remainders, he obtains the optimal estimate

$$(1.3) \quad \|\mathbf{u} - \mathbf{u}^N\|_{L^2} + \varepsilon^{1/2} \left\| \left(\pi_{1/2}^\varepsilon \nabla \right) (\mathbf{u} - \mathbf{u}^N) \right\|_{L^2} + \|p - \varepsilon^{-1} p^N\|_{L^2} \leq C \varepsilon^{N+1}.$$

Nazarov's paper is concise to the point of being impenetrable at times. We interpret $\pi_{1/2}^\varepsilon \nabla = (\partial_x, \partial_y, \varepsilon^{1/2} \partial_z)$, but this symbol was not defined and may actually be a variable coefficient operator that incorporates the boundary conditions in its definition. We are also unsure of the definition of p and p^N , as we would have expected $p - \varepsilon^{-2} p^N$ to appear together.

In a later paper [19], Nazarov studies the asymptotics of the solution of the Stokes equations in a domain in which two smooth surfaces meet at a point. This problem is also studied in a recent paper of Ciuperca, Hafidi, and Jai in [9]. This singular limit is interesting in that deriving even the first correction to the zeroth order approximation in the asymptotic expansion remains an open problem.

Assemien, Bayada, and Chambat [2] have studied the important question of the effect of inertia on the asymptotic behavior of a thin film flow, which can in many cases be significant, requiring that the Navier–Stokes equations be used in place of the Stokes equations as the underlying model for the asymptotic expansion. We also mention that there is a large body of literature on the long-time behavior of solutions of the Navier–Stokes equations on thin domains; see, e.g., [21, 17].

In 2000, Duvnjak and Marušić-Paloka [11] showed how to rigorously analyze the lubrication approximation of the Navier–Stokes equations for a slipper bearing in a circular geometry. The focus of their paper is on formulating the problem in cylindrical coordinates and showing how to adapt the zeroth order case of Nazarov's proof to handle the change of variables. Elrod's pioneering 1960 paper [12] is also concerned with the (formal) relationship between the Navier–Stokes equations and Reynolds' equation for this geometry.

1.2. Motivation and summary. None of the studies described above shows how the constant C bounding the error depends on the function $h(x)$ describing the geometry. This is because most theorems of analysis give constants that depend on the domain Ω , which is usually fixed. But in our case, the data $h(x)$ of the problem actually specifies the domain; therefore, to obtain bounds that are independent of h , one must avoid or modify standard arguments for flattening the boundary, etc., so as not to lose track of $h(x)$ in the analysis. Moreover, arguments based on the closed graph theorem or Rellich's compactness theorem must be avoided entirely, as these also depend on the geometry. This forces us to look for new ways to analyze old problems using tools that furnish explicit constants.

In this paper, we consider only the two-dimensional, periodic Stokes equations with a specific choice of boundary conditions, but we derive error estimates that depend on h in an explicit, intuitive way. Our main result is summarized in Theorem 4.11, which may be stated as follows: Let $T = [0, 1]_p$ be the periodic unit interval. If $k \geq 0$, $h \in C^{2k+1,1}(T)$, $0 < h_0 \leq h(x) \leq 1$ for $x \in T$, and $\varepsilon \leq r_0/3$ (defined below), then the error in truncating the expansion of the stream function, velocity, vorticity, and pressure (in appropriate ε -weighted Sobolev norms) at order $2k$ (keeping in mind that only even powers of ε appear in these expansions) is bounded by

$$(1.4) \quad \sqrt{I_1} (|V_0| + |V_1|) \left[1 + \theta_k \frac{\varepsilon}{r_k} \sqrt{\frac{I_3}{I_1}} \right] \left(\frac{\varepsilon}{\rho_k r_k} \right)^{2k+2},$$

where V_0 and V_1 are prescribed tangential velocities on the lower and upper boundaries of the domain

$$(1.5) \quad r_k = \left(\max_{1 \leq \ell \leq 2k+2} \left\{ \left\| \frac{1}{\ell!} h^{\ell-1} \partial_x^\ell h \right\|_\infty^{1/\ell} \right\} \right)^{-1}, \quad I_m = \int_0^1 h(x)^{-m} dx$$

and ρ_k, θ_k are constants independent of h . The bound on pressure has another term involving h_0 ; see (4.110) below.

The constants in (1.4) have been divided into two types: those that are (1) given in the problem statement or easily computable from h ; or (2) difficult to compute but universal (independent of h). We show how to compute the constants in the latter category (ρ_k and θ_k) in section 4; see Table 4.4. The constants in the former category (r_k and I_m) help us understand the competing mechanism of singularity formation versus proximity to the substrate: the curvature and higher derivatives are allowed to diverge as long as the gap size simultaneously approaches zero in such a way that the homogeneous products $\frac{1}{\ell!} h^{\ell-1} \partial_x^\ell h$ remain uniformly bounded. Although the factors $\sqrt{I_1}$ and $\sqrt{I_3/I_1}$ in (1.4) also diverge in this limit, the norm of the exact solution diverges at a similar rate — so the relative error in the expansion solution truncated at order $2k$ is $O(\varepsilon^{2k+2})$, with $\rho_k r_k$ serving as an effective radius of convergence.

The framework we have chosen for this paper is intended to be general enough to cover many interesting applications (such as a crawling gastropod [25] or an “unwrapped” slipper bearing) but simple enough to obtain explicit detailed estimates that reveal the dependence of the error on the geometry $h(x)$. We also wanted to determine whether there might exist geometries for which the asymptotic expansion yields a convergent series. Although we do not have a rigorous proof, the answer appears to be negative even for the simplest case of a real analytic function such as $h(x) = \frac{3}{5} + \frac{2}{5} \sin 2\pi x$, for which the r_k in (1.5) are bounded away from zero. It is hoped that this work will serve as a useful first step toward obtaining similar error estimates for three-dimensional problems that include more general boundary conditions, incorporate end effects near the lateral edges of the domain (which we avoid by studying the periodic case), and include the effect of inertia or viscoelasticity.

1.3. Outline. In section 2, we derive Reynolds’ lubrication approximation in its primitive and stream function formulations. In section 3, we show how to compute successive terms in an asymptotic expansion of the stream function. In section 3.2, we prove a structure theorem describing the dependence of these terms on $h(x)$ and its derivatives.

In section 4, we formulate the problem weakly and analyze the truncation error equation using weighted Sobolev spaces and a uniform Poincaré–Friedrichs argument. The first challenge is to find the right weighted norms on the lower and upper boundaries (equivalent to $H^{1/2}(\Gamma_0)$ and $H^{1/2}(\Gamma_1)$ for fixed ε) to yield manageable error estimates in terms of h when we change variables to straighten out the boundaries. In section 4.4, we reduce the problem of bounding the truncation errors to that of bounding the second and fourth derivatives of the two highest order terms retained in the asymptotic expansion, namely, $\|\psi_{xx}^{(2k)}\|_0$ and $\|h^2 \psi_{xxxx}^{(2k-2)}\|_0$. We then use the structure theorem of section 3.2 to compute these norms in order to obtain the constants ρ_k and θ_k in (1.4) for $0 \leq k \leq 25$. In section 4.5, we show how to compute the error in velocity, vorticity, and pressure from that of the stream function. This requires that we determine how the Babuška–Brezzi inf-sup constant β depends on $h(x)$; see [24].

In section 5, we validate our results by comparing to “exact” solutions (computed using finite elements) for a geometry typical of engineering applications. The result of this comparison is that the effective radius of convergence $r_k \rho_k$ is within a factor of 3 of optimal for $k = 5$, $k = 10$, and perhaps all $k \geq 5$. These calculations also suggest that even when $h(x)$ is real analytic, the expansion solution is an asymptotic series rather than a convergent series. This is because the constants ρ_k converge to zero as $k \rightarrow \infty$. Fortunately, ρ_k initially increases and does not become smaller than $\rho_0 = 0.197$ until $2k = 26$, which is already outside of the practical range of k . Finally, in Appendix A, we present our numerical algorithm for computing the expansion solutions, which can be performed symbolically using a computer algebra system such as Mathematica or in floating point arithmetic, e.g., in C^{++} .

2. Reynolds' approximation. Consider the Stokes equations on a periodic domain of width \bar{W} bounded below by a flat wall moving with constant speed \bar{V}_0 and above by an inextensible sheet moving with constant speed \bar{V}_1 along a fixed curve $\Gamma_{1,\varepsilon} = \{(\bar{x}, \bar{h}(\bar{x})) : 0 \leq \bar{x} \leq \bar{W}\}$; see Figure 2.1. A bar is used to distinguish a physical variable from its dimensionless counterpart. We nondimensionalize the variables by choosing a characteristic speed \bar{U} and height \bar{H} for the problem, and set $\bar{x} = \bar{W}x$, $\bar{y} = \bar{H}y$, $\bar{h}(\bar{x}) = \bar{H}h(x)$, $\bar{V}_i = \bar{U}V_i$, $(\bar{u}, \bar{v}) = \bar{\mathbf{u}} = (\bar{U}u, \bar{U}\frac{\bar{H}}{\bar{W}}v)$, and $\bar{p} = \bar{\mu}\frac{\bar{U}\bar{W}}{\bar{H}^2}p$. The stream function $\bar{\psi}$, flux \bar{Q} , and vorticity $\bar{\omega}$ introduced below satisfy $\bar{\psi} = \bar{U}\bar{H}\psi$, $\bar{Q} = \bar{U}\bar{H}Q$, and $\bar{\omega} = \frac{\bar{U}}{\bar{H}}\omega$.

We have in mind a situation where the aspect ratio $\varepsilon = \bar{H}/\bar{W}$ of the physical domain is small. By scaling the x - and y -axes differently, we map the problem onto a nicer geometry, which introduces terms in the equations that vanish in the singular limit $\varepsilon \rightarrow 0$. Specifically, we wish to find x -periodic functions u, v, p defined on the rescaled domain

$$(2.1) \quad \Omega = \{(x, y) : 0 \leq x \leq 1, \ 0 < y < h(x)\}$$

such that

$$(2.2) \quad p_x = \varepsilon^2 u_{xx} + u_{yy}, \quad p_y = \varepsilon^4 v_{xx} + \varepsilon^2 v_{yy}, \quad v_y = -u_x \quad (\text{in } \Omega)$$

subject to periodic boundary conditions on the left and right sides of Ω and

$$(2.3) \quad (u, v)|_{\Gamma_0} = (g_0, 0), \quad (u, v)|_{\Gamma_1} = (g_1, h_x g_1)$$

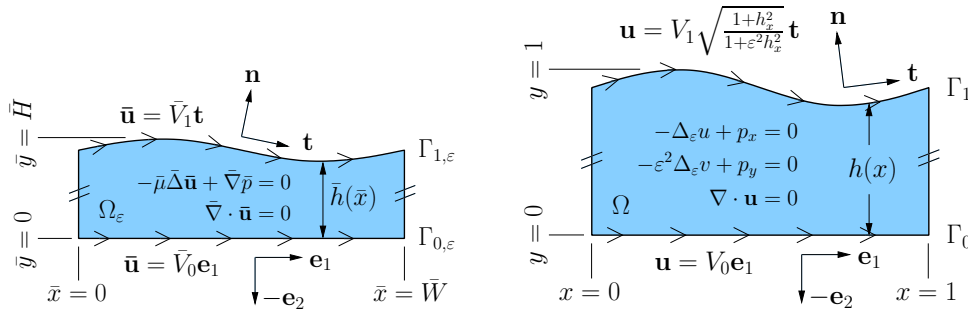


FIG. 2.1. Geometry commonly encountered in lubrication-type problems. Left: Physical coordinate system. Right: Dimensionless coordinate system ($\Delta_\varepsilon = \varepsilon^2 \partial_x^2 + \partial_y^2$).

on the bottom and top boundaries. Here

$$(2.4) \quad g_0(x) = V_0, \quad g_1(x) = V_1 [1 + \varepsilon^2 h'(x)^2]^{-1/2},$$

i.e., $g_1(x) = V_1 \cos \theta(x)$, where $\theta = \arctan(\varepsilon h_x)$ is the angle of the curve $\bar{h}(\bar{x})$ relative to the horizontal. Reynolds' lubrication approximation is obtained by setting $\varepsilon = 0$ in the equations and solving

$$(2.5) \quad p_x = u_{yy}, \quad p_y = 0, \quad v_y = -u_x, \quad \mathbf{u}|_{\Gamma_0} = (V_0; 0), \quad \mathbf{u}|_{\Gamma_1} = (1; h_x)V_1.$$

If we write (2.2) in the form $L(\mathbf{u}; p) = (0; 0; 0)$, where $L = L^{(0)} + \varepsilon^2 L^{(2)} + \varepsilon^4 L^{(4)}$ is given by

$$(2.6) \quad L = \begin{pmatrix} -\partial_y^2 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -\partial_x^2 & 0 & 0 \\ 0 & -\partial_y^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^4 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\partial_x^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then (2.5) is just the zeroth order system $L^{(0)}(\mathbf{u}; p) = (0; 0; 0)$ with zeroth order boundary conditions (expanding g_0 and g_1 in (2.3) in powers of ε). The equation for v decouples from the others, and we find that p is independent of y and

$$(2.7) \quad u(x, y) = \left(\frac{y^2}{2} - \frac{h(x)y}{2} \right) p_x(x) + \left(1 - \frac{y}{h(x)} \right) V_0 + \frac{y}{h(x)} V_1.$$

Integrating from 0 to h and solving for p_x , we obtain

$$(2.8) \quad p_x = \frac{6}{h^2}(V_0 + V_1) - \frac{12}{h^3}Q,$$

where $Q = \int_0^h u(x, y) dy$ is the volume flux through any cross section of the fluid. (Q is constant since $\nabla \cdot \mathbf{u} = 0$ and \mathbf{u} is tangent to Γ_0 and Γ_1). Since p is periodic, $\int p_x dx = 0$, and we find that

$$(2.9) \quad Q = \frac{V_0 + V_1}{2} \frac{I_2}{I_3} \quad \left(I_m = \int_0^1 h(x)^{-m} dx \right).$$

Substituting (2.9) and (2.8) into (2.7) and using $v_y = -u_x$, $v(x, 0) = 0$, we obtain the solution

$$(2.10) \quad \begin{aligned} p_x &= \frac{6(V_0 + V_1)}{h^2} \left(1 - \frac{I_2}{I_3 h} \right), \\ u &= (V_0 + V_1) \left(3 \frac{I_2}{I_3 h} - 3 \right) \left(\frac{y}{h} - \frac{y^2}{h^2} \right) + \left(1 - \frac{y}{h} \right) V_0 + \frac{y}{h} V_1, \\ v &= (V_0 + V_1) \left(3 \frac{I_2}{I_3 h} - 2 \right) \left(\frac{y^2}{h^2} - \frac{y^3}{h^3} \right) h_x + V_1 \frac{y^2}{h^2} h_x. \end{aligned}$$

The vertical component v of the velocity field is customarily omitted from zeroth order lubrication theory as $\bar{v} = \varepsilon \bar{U} v$ is $O(\varepsilon)$ on the thin geometry Ω_ε of Figure 2.1.

We may also derive (2.10) using a stream function formulation of the problem. Our procedure for computing higher order corrections to the lubrication approximation and our method for estimating the error of these expansion solutions are both done in the stream function formulation. Let us define

$$(2.11) \quad \Delta_\varepsilon = \varepsilon^2 \partial_x^2 + \partial_y^2, \quad \mathbf{u}_\varepsilon = \begin{pmatrix} u \\ \varepsilon^2 v \end{pmatrix}.$$

In our error estimates below, we will need to consider the inhomogeneous problem $L(\mathbf{u}; p) = (F_1; F_2; 0)$ with boundary conditions (2.3), i.e.,

$$(2.12) \quad \begin{aligned} -\Delta_\varepsilon \mathbf{u}_\varepsilon + \nabla p &= \mathbf{F}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned} \quad \mathbf{u}|_{\Gamma_0} = (g_0; 0), \quad \mathbf{u}|_{\Gamma_1} = (g_1; h_x g_1).$$

Since \mathbf{u} is incompressible, there is a stream function ψ such that

$$(2.13) \quad \mathbf{u} = \nabla \times \psi = (\psi_y, -\psi_x), \quad \nabla \times \mathbf{u}_\varepsilon = \varepsilon^2 v_x - u_y = -\Delta_\varepsilon \psi.$$

It follows from (2.12) that ψ satisfies the rescaled biharmonic equation

$$(2.14) \quad \Delta_\varepsilon^2 \psi = \psi_{yyyy} + 2\varepsilon^2 \psi_{xxyy} + \varepsilon^4 \psi_{xxxx} = \nabla \times \mathbf{F},$$

with periodic boundary conditions in the x -direction and

$$(2.15) \quad \left\{ \begin{array}{l} \psi = 0 \\ \psi_y = g_0 \end{array} \right\} \text{ on } \Gamma_0, \quad \left\{ \begin{array}{l} \psi = Q \\ \psi_y(x, h(x)) = g_1 \end{array} \right\} \text{ on } \Gamma_1,$$

where $Q = \int_0^{h(0)} u(0, y) dy$. Since p is periodic, $\int_0^1 p_x(x, 0) dx = 0$, i.e.,

$$(2.16) \quad \int_0^1 \psi_{yyy}(x, 0) + F_1(x, 0) dx = 0.$$

Conversely, suppose we are able to find a flux Q and a classical solution ψ of (2.14) and (2.15) such that (2.16) holds. Then we define $\mathbf{u} = \nabla \times \psi$ and note that (2.14) implies $\nabla \times (\Delta_\varepsilon \mathbf{u}_\varepsilon + \mathbf{F}) \equiv 0$, i.e., the integral

$$(2.17) \quad p(x, y) = \int_\gamma (\Delta_\varepsilon \mathbf{u}_\varepsilon + \mathbf{F}) \cdot \mathbf{t} ds, \quad \left(\begin{array}{l} \mathbf{t} = \text{unit tangent vector along} \\ \text{path } \gamma \text{ joining } (0, 0) \text{ to } (x, y) \end{array} \right)$$

is independent of the path γ . A canonical choice for γ is

$$(2.18) \quad p(x, y) = \int_0^x [\varepsilon^2 u_{xx} + u_{yy} + F_1](\xi, 0) d\xi + \int_0^y [\varepsilon^4 v_{xx} + \varepsilon^2 v_{yy} + F_2](x, \eta) d\eta.$$

Condition (2.16) is equivalent to requiring $p(1, 0) = p(0, 0)$, from which it follows that $p(1, y) = p(0, y)$ for $0 \leq y \leq h(0)$, since the integrand of the second integral in (2.18) is periodic in x . By construction, the variables \mathbf{u}, p satisfy (2.12), where the boundary condition on Γ_1 follows from the fact that $\psi_x + h_x \psi_y = 0$ there; hence, classical solutions of the rescaled biharmonic equation yield classical solutions of the rescaled Stokes equations and vice versa. Reynolds' approximation (2.10) is recovered if \mathbf{F} and ε are set to zero in (2.14)–(2.16) when solving for ψ and Q ; see section 3.1.

3. Higher order corrections. In this section we show how to compute successive terms in the formal expansion of the solution of the rescaled biharmonic equation (2.14) in powers of $\varepsilon = \bar{H}/\bar{W}$. For this purpose, it is convenient to manipulate the equations assuming they are satisfied classically. Once we obtain formulas for the higher order approximations, we will show (in section 4) that they satisfy a weak formulation of the problem that makes it possible to obtain error estimates. See [15] for background on perturbation methods in partial differential equations.

3.1. A recursive algorithm. Matching like powers of ε in the expansion

$$(3.1) \quad [\partial_y^4 + 2\varepsilon^2 \partial_x^2 \partial_y^2 + \varepsilon^4 \partial_x^4] [\psi^{(0)} + \varepsilon^2 \psi^{(2)} + \varepsilon^4 \psi^{(4)} + \dots] = 0,$$

we obtain the recursion

$$(3.2) \quad \begin{aligned} \psi_{yyyy}^{(0)} &= 0, \\ \psi_{yyyy}^{(2)} &= -2\psi_{xxyy}^{(0)}, \\ \psi_{yyyy}^{(2k)} &= -2\psi_{xxyy}^{(2k-2)} - \psi_{xxxx}^{(2k-4)}, \end{aligned} \quad k = 2, 3, 4, \dots$$

The boundary conditions (2.15) become

$$(3.3) \quad B\psi^{(2k)} = \left(0, g_0^{(2k)}, Q^{(2k)}, g_1^{(2k)}\right), \quad k = 0, 1, 2, 3, \dots,$$

where $B\psi = (\psi|_{\Gamma_0}, \psi_y|_{\Gamma_0}, \psi|_{\Gamma_1}, \psi_y|_{\Gamma_1})$ and $g_0(x), g_1(x)$ were defined in (2.4):

$$(3.4) \quad g_0^{(2k)}(x) = \begin{cases} V_0, & k = 0, \\ 0, & k > 0, \end{cases} \quad g_1^{(2k)}(x) = V_1 \binom{-1/2}{k} h'(x)^{2k}.$$

Condition (2.16) (with $F_1 = 0$) becomes

$$(3.5) \quad \int_0^1 \psi_{yyy}^{(2k)}(x, 0) dx = 0, \quad k = 0, 1, 2, \dots$$

If \mathbf{F} were nonzero in (2.14) and depended on ε in such a way that $\nabla \times \mathbf{F}$ had an expansion in even powers of ε , we could incorporate these terms into (3.2) and (3.5) as well; however, we will assume $\mathbf{F} = \mathbf{0}$ except in section 4, where we consider the general case only to derive error estimates for the $\mathbf{F} = \mathbf{0}$ case. Let us denote the right-hand side of (3.2) by $f^{(2k)}(x, y)$ for $k \geq 0$. The terms $\psi^{(2k)}, Q^{(2k)}$ in (3.2) and (3.3) may be computed via

$$(3.6) \quad \left(\psi^{(2k)}, Q^{(2k)}\right) = G\left(f^{(2k)}, g_0^{(2k)}, g_1^{(2k)}\right), \quad k = 0, 1, 2, \dots,$$

where G is defined by Algorithm 3.1 in Figure 3.1. In this algorithm, we solve $\psi_{yyyy} = f$ by integrating four times in the y -direction and then correct the boundary conditions with a cubic polynomial. The formula for Q in the algorithm may be derived from the one for ψ as follows. As $\psi_{0,yyy}(x, 0) = 0$, the requirement that $\int_0^1 \psi_{yyy}(x, 0) dx = 0$ is equivalent to the condition

$$(3.7) \quad 0 = 6 \int_0^1 \frac{-2Q + 2\psi_0 - \psi_{0,y}h + g_0h + g_1h}{h^3} dx.$$

Solving for Q and using $\int h^{-3} dx = I_3$ gives the result.

The formulas $(u, v) = (\psi_y, -\psi_x)$, $\omega = \varepsilon^2 v_x - u_y$, $p_x = u_{yy} + \varepsilon^2 u_{xx}$, and $p_y = \varepsilon^2 v_{yy} + \varepsilon^4 v_{xx}$ allow us to compute the expansions of \mathbf{u} , ω , and p in terms of ψ :

$$(3.8) \quad \begin{aligned} u^{(2k)} &= \psi_y^{(2k)}, & v^{(2k)} &= -\psi_x^{(2k)}, & k &\geq 0, \\ \omega^{(0)} &= -\psi_{yy}^{(0)}, & \omega^{(2k)} &= -\psi_{xx}^{(2k-2)} - \psi_{yy}^{(2k)}, & k &\geq 1, \\ p_x^{(0)} &= \psi_{yyy}^{(0)}, & p_x^{(2k)} &= \psi_{xxy}^{(2k-2)} + \psi_{yyy}^{(2k)}, & k &\geq 1, \\ p_y^{(0)} &= 0, & p_y^{(2)} &= -\psi_{xyy}^{(0)}, & p_y^{(2k)} &= -\psi_{xxx}^{(2k-4)} - \psi_{xyy}^{(2k-2)}, & k &\geq 2, \\ p^{(2k)}(x, y) &= \int_0^x p_x^{(2k)}(\xi, 0) d\xi + \int_0^y p_y^{(2k)}(x, \eta) d\eta, & k &\geq 0. \end{aligned}$$

ALGORITHM 3.1. $(\psi, Q) = G(f, g_0, g_1)$:

$$\begin{aligned} \psi_0 &= \mathbb{V}^4 f \quad \left(\mathbb{V} = \text{Volterra operator: } \mathbb{V}f(x, y) = \int_0^y f(x, \eta) d\eta \right) \\ Q &= \frac{1}{2I_3} \int_0^1 \frac{2\psi_0(x, h(x))}{h(x)^3} + \frac{-\psi_{0,y}(x, h(x)) + g_0 + g_1(x)}{h(x)^2} dx \\ \psi(x, y) &= \psi_0(x, y) + (g_0 h(x)) \frac{y}{h(x)} \\ &\quad + (3Q - 3\psi_0(x, h(x)) + \psi_{0,y}(x, h(x))h(x) - 2g_0 h(x) - g_1(x)h(x)) \frac{y^2}{h(x)^2} \\ &\quad + (-2Q + 2\psi_0(x, h(x)) - \psi_{0,y}(x, h(x))h(x) + g_0 h(x) + g_1(x)h(x)) \frac{y^3}{h(x)^3} \\ \text{return } &(\psi, Q) \end{aligned}$$

FIG. 3.1. Algorithm to solve $\psi_{yyyy} = f$, $B\psi = (0, g_0, Q, g_1)$, $\int_0^1 \psi_{yyy}(x, 0) dx = 0$.

Equation (3.2) implies that $\partial_x p_y^{(2k)} = \partial_y p_x^{(2k)}$ for $k \geq 0$; hence, differentiating under the integral sign in (3.8), we see that $p_x^{(2k)}$ and $p_y^{(2k)}$ actually are the partial derivatives of $p^{(2k)}$. Finally, our choice of $Q^{(2k)}$ ensures $\int_0^1 p_x^{(2k)}(\xi, 0) d\xi = \int_0^1 \psi_{yyy}^{(2k)}(x, 0) dx = 0$ so that $p^{(2k)}$ is periodic.

Using Algorithm 3.1 to evaluate $(\psi^{(0)}, Q^{(0)}) = G(0, V_0, V_1)$ yields

$$(3.9) \quad Q^{(0)} = \frac{V_0 + V_1}{2} \frac{I_2}{I_3},$$

$$(3.10) \quad \psi^{(0)} = (V_0 h) \frac{y}{h} + \left(3Q^{(0)} - (2V_0 + V_1)h \right) \frac{y^2}{h^2} + \left(-2Q^{(0)} + (V_0 + V_1)h \right) \frac{y^3}{h^3},$$

which agrees with Reynolds' approximation (2.10) when $\mathbf{u}^{(0)}, p^{(0)}$ are computed from $\psi^{(0)}$. To compute higher order terms in the expansion, we need to study the recursion (3.6) more closely to determine how h will enter into the formulas for $Q^{(2k)}$ and $\psi^{(2k)}$.

3.2. Algebraic structure of the stream function expansion. In this section, we show how the terms $\psi^{(2k)}$ and $Q^{(2k)}$ in the stream function expansion depend on h . The key result of this section is that these higher order corrections have a structure similar to the zeroth order formulas (3.9) and (3.10), but the coefficient on each $\frac{y^n}{h^n}$ now belongs to a more complicated polynomial algebra in the symbols V_0, V_1, h , the derivatives of h , and certain weighted averages of the products of h and its derivatives. We also present a concise representation for the correction terms using matrices of rational numbers that are independent of any particular choice of shape function h . By splitting the analysis into one part that holds universally and another that depends on h in a simple way, we are able to derive useful error estimates governing the expansion solution truncated at any order in section 4.

Let $\mathcal{P} = \mathbb{Q}[h, h_x, h_{xx}, \dots]$ denote the algebra of polynomials in h and its derivatives over the rationals. A typical element of \mathcal{P} might be $3 + \frac{2}{5}h^2 h_{xx} h_{xxx}^3$. In \mathcal{P} , the generators h, h_x , etc., are treated as symbols rather than functions. Thus, if $h(x)$ happens to equal 1 identically, the polynomials $1 - h$ and h_x^6 are nonzero in \mathcal{P} even though they are mapped to zero when \mathcal{P} is (noninjectively) embedded in $C^\infty(T)$, the space of C^∞ functions on the periodic interval $T = [0, 1]_p$. If h is not smooth, its derivatives can still be manipulated symbolically and various subspaces (involving

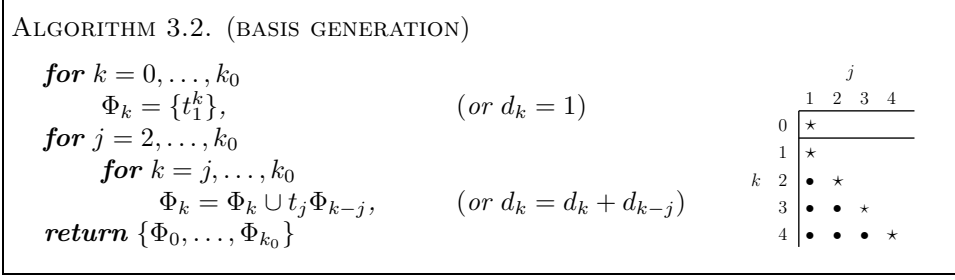


FIG. 3.2. Algorithm to find a canonical basis Φ_k for each space \mathcal{H}_k in the range $0 \leq k \leq k_0$. Here $t_1 \leftrightarrow h_x, \dots, t_k \leftrightarrow \frac{1}{k!} h^{k-1} \partial_x^k h$.

terms with few enough derivatives) can still be embedded in actual function spaces such as $C^k(T)$.

For any monomial $\alpha = C h^{i_0} h_x^{i_1} h_{xx}^{i_2} \dots \in \mathcal{P}$ with $C \neq 0$, we define its *superdegree* to be the number of derivatives present:

$$(3.11) \quad \text{sdeg}(\alpha) = i_1 + 2i_2 + 3i_3 + \dots$$

If $\alpha \in \mathcal{P}$, we define its superdegree to be the maximal superdegree of any of its terms, and set $\text{sdeg}(0) = -\infty$. Since \mathbb{Q} is a field, $\text{sdeg}(\alpha\beta) = \text{sdeg}(\alpha) + \text{sdeg}(\beta)$ for any $\alpha, \beta \in \mathcal{P}$. We say that α is homogeneous of superdegree k if each of its terms has superdegree k .

Let $\mathcal{H} \subset \mathcal{P}$ denote the subalgebra generated by the set $\{h^{k-1} \partial_x^k h : k \geq 1\}$, i.e.,

$$(3.12) \quad \mathcal{H} = \mathbb{Q}[\{h_x, h h_{xx}, h^2 h_{xxx}, \dots\}],$$

and for $k \geq 0$, let $\mathcal{H}_k \subset \mathcal{H}$ denote the subspace

$$(3.13) \quad \mathcal{H}_k = \{0\} \cup \{\alpha \in \mathcal{H} : \alpha \text{ is homogeneous of superdegree } k\}.$$

Note that \mathcal{H}_k is finite-dimensional for all k , and $\mathcal{H}_0 = \mathbb{Q}$ is the set of constant polynomials. We will denote the dimension of \mathcal{H}_k by

$$(3.14) \quad d_k = \dim(\mathcal{H}_k).$$

Given an integer $k_0 \geq 0$, we can use Algorithm 3.2 in Figure 3.2 to construct a canonical basis $\Phi_k = \{\varphi_1^{(k)}, \dots, \varphi_{d_k}^{(k)}\}$ for each \mathcal{H}_k with k in the range $0 \leq k \leq k_0$. For notational convenience, let t_j stand for $\frac{1}{j!} h^{j-1} \partial_x^j h$. As the outer loop (on j) progresses, Φ_k contains a basis for the subspace of \mathcal{H}_k that involves only the symbols t_1, \dots, t_j . Let us denote these auxiliary sets by

$$(3.15) \quad \Phi_{kj} = \{t_1^{i_1} \dots t_j^{i_j} : i_1 + 2i_2 + \dots + j i_j = k\}, \quad 1 \leq j \leq k.$$

Then $\Phi_{k1} = \{t_1^k\}$, $\Phi_{kk} = \Phi_k$, and $\Phi_{kj} = \Phi_{k,j-1} \cup t_j \Phi_{k-j,j}$ for $2 \leq j \leq k$. In other words, Φ_{kj} consists of $\Phi_{k,j-1}$ together with all products of the variables t_1, \dots, t_j of superdegree k that contain at least one power of t_j . The first several Φ_k and d_k are

given by

(3.16)

$$\begin{aligned}\Phi_0 &= \{1\}, \quad \Phi_1 = \{h_x\}, \quad \Phi_2 = \left\{h_x^2, \frac{hh_{xx}}{2}\right\}, \quad \Phi_3 = \left\{h_x^3, \frac{hh_x h_{xx}}{2}, \frac{h^2 h_{xxx}}{6}\right\}, \\ \Phi_4 &= \left\{h_x^4, \frac{hh_x^2 h_{xx}}{2}, \frac{h^2 h_{xx}^2}{4}, \frac{h^2 h_x h_{xxx}}{6}, \frac{h^3 h_{xxxx}}{24}\right\}, \\ (d_0, \dots, d_{10}) &= \{1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42\}, \quad d_{20} = 627, \quad d_{50} = 204226.\end{aligned}$$

We have found empirically that the first 75000 terms satisfy $\frac{1}{2}(\frac{13^{\sqrt{k}}}{6k+1}) < d_k < \frac{13^{\sqrt{k}}}{6k+1}$. In fact, we have recently learned of the Hardy–Ramanujan formula

$$(3.17) \quad d_k \sim \frac{\exp\left(\pi\sqrt{2k/3}\right)}{4k\sqrt{3}} \quad \text{as } k \rightarrow \infty$$

for the number of partitions of the integer k . Thus, rather than 13, the base is in fact $e^{\pi\sqrt{2/3}} = 13.001954$.

We can now describe the structure of the stream function expansion in terms of the shape function h . In the following theorem, $\mathcal{V}_1 \mathcal{H}_{2k}$ is the tensor product of \mathcal{V}_1 and \mathcal{H}_{2k} , where

$$(3.18) \quad \mathcal{V}_1 = \{0\} \cup \{\alpha \in \mathbb{Q}[V_0, V_1] : \alpha \text{ is homogeneous of degree } 1\}$$

is the space of rational linear combinations of V_0 and V_1 . Recall from (2.9) above that $I_m = \int_0^1 h(x)^{-m} dx$.

THEOREM 3.3. *The terms $Q^{(2k)}, \psi^{(2k)}$ in the stream function expansion defined by the recursion (3.6) and Algorithm 3.1 have the form*

(3.19)

$$Q^{(2k)} = \frac{I_2}{I_3} a^{(2k)} + \sum_{\ell=0}^{k-1} Q^{(2\ell)} b^{(2k-2\ell)}, \quad \psi^{(2k)} = \frac{I_2}{I_3} \alpha^{(2k)} + \sum_{\ell=0}^k Q^{(2\ell)} \beta^{(2k-2\ell)},$$

where

$$(3.20) \quad \alpha^{(2k)}(x, y) = \sum_{n=1}^{2k+3} \alpha_n^{(2k)}(x) \frac{y^n}{h(x)^n}, \quad \beta^{(2k)}(x, y) = \sum_{n=1}^{2k+3} \beta_n^{(2k)}(x) \frac{y^n}{h(x)^n},$$

and

$$(3.21) \quad \alpha_n^{(2k)} \in \frac{I_3}{I_2} h \mathcal{V}_1 \mathcal{H}_{2k}, \quad \beta_n^{(2k)} \in \mathcal{H}_{2k}.$$

Moreover, $a^{(2k)} = \frac{1}{2I_3} \int_0^1 \frac{\alpha_3^{(2k)}(x)}{h(x)^3} dx$ and $b^{(2k)} = \frac{1}{2I_3} \int_0^1 \frac{\beta_3^{(2k)}(x)}{h(x)^3} dx$.

Remark 3.4. In addition to pinning down the way in which h appears in the formulas for the stream function expansion, this theorem allows us to represent $\psi^{(2k)}$ and $Q^{(2k)}$ using matrices of rational numbers. Explicitly, (3.20) and (3.21) hold iff there are matrices $A^{(2k)}, B^{(2k)}$ with entries in \mathcal{V}_1 and \mathbb{Q} , respectively, with rows indexed from 0 to $(2k+3)$ and columns indexed from 1 to d_{2k} , and containing

only zeros in row 0, such that

$$(3.22) \quad \begin{aligned} \alpha^{(2k)}(x, y) &= (Y_{2k}(x, y))^T A^{(2k)} \left(\frac{I_3}{I_2} h(x) \Phi_{2k}(x) \right), \\ \beta^{(2k)}(x, y) &= (Y_{2k}(x, y))^T B^{(2k)} \Phi_{2k}(x), \end{aligned}$$

where $Y_{2k} = (1, \frac{y}{h}, \dots, (\frac{y}{h})^{2k+3})^T$ and $\Phi_{2k} = (\varphi_1^{(2k)}, \dots, \varphi_{d_{2k}}^{(2k)})^T$ are treated as column vectors. The purpose of the zeroth row is to make it easy to convert to orthogonal polynomials in y/h if desired. The final statement of the theorem asserts that the formulas for $a^{(2k)}$ and $b^{(2k)}$ are also encoded in the matrices $A^{(2k)}$ and $B^{(2k)}$. If we adopt Matlab notation and denote row i of $A^{(2k)}$ by $A^{(2k)}(i, :)$, then

$$(3.23) \quad a^{(2k)} = \frac{1}{2} A^{(2k)}(3, :) E_2^{(2k)}, \quad b^{(2k)} = \frac{1}{2} B^{(2k)}(3, :) E_3^{(2k)},$$

where

$$(3.24) \quad E_m^{(2k)} = \left(E_{m,1}^{(2k)}, \dots, E_{m,d_{2k}}^{(2k)} \right)^T = \frac{1}{I_m} \int_0^1 \frac{\Phi_{2k}(x)}{h(x)^m} dx.$$

Note that $E_{m,j}^{(2k)}$ is the weighted average of $\varphi_j^{(2k)}$ with weight function $I_m^{-1} h^{-m}$. For example, $E_m^{(0)} = (1)$, $E_m^{(2)} = (\frac{1}{I_m} \int_0^1 \frac{h^2}{h^m} dx, \frac{1}{I_m} \int_0^1 \frac{hh_{xx}}{2h^m} dx)^T$, etc.; see (3.16) above.

Example 3.5. We can now represent $Q^{(0)}$ and $\psi^{(0)}$ in (3.9) and (3.10) by

$$(3.25) \quad a^{(0)} = \frac{V_0 + V_1}{2}, \quad A^{(0)} = V_0 \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} + V_1 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad B^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -2 \end{pmatrix}.$$

The second order terms $Q^{(2)}$ and $\psi^{(2)}$ involve these as well as

$$(3.26) \quad a^{(2)} = \frac{1}{2} [V_0 \left(\frac{7}{15}, \frac{2}{15} \right) + V_1 \left(\frac{19}{30}, -\frac{8}{15} \right)] E_2^{(2)}, \quad b^{(2)} = \frac{1}{2} \left(-\frac{6}{5}, -\frac{2}{5} \right) E_3^{(2)},$$

$$A^{(2)} = V_0 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -8/15 & 2/15 \\ 7/15 & 2/15 \\ 2/3 & -2/3 \\ -3/5 & 2/5 \end{pmatrix} + V_1 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -11/30 & 7/15 \\ 19/30 & -8/15 \\ 1/3 & -1/3 \\ -3/5 & 2/5 \end{pmatrix}, \quad B^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 9/5 & -2/5 \\ -6/5 & -2/5 \\ -3 & 2 \\ 12/5 & -6/5 \end{pmatrix}.$$

For $k \geq 2$, $A^{(2k)}$ and $B^{(2k)}$ are both $(2k+4) \times d_{2k}$ matrices with rows 0 and 1 containing only zeros. These matrices are universal: the shape function h enters into the formulas only through Y_{2k} , Φ_{2k} , and $E_m^{(2k)}$ in (3.22) and (3.23). In Appendix A, we show how to compute $A^{(2k)}$ and $B^{(2k)}$ directly from the lower order matrices $A^{(2\ell)}$ and $B^{(2\ell)}$ with $0 \leq \ell < k$.

Proof of Theorem 3.3: We saw in Example 3.5 above that $Q^{(0)}$ and $\psi^{(0)}$ have the desired form. Suppose $k_0 \geq 1$, and the theorem holds for $0 \leq k < k_0$. We must show that it is also true for $k = k_0$. By (3.6),

$$(3.27) \quad \left(\psi^{(2k_0)}, Q^{(2k_0)} \right) = \begin{cases} G \left(-2\psi_{xyxy}^{(2k_0-2)}, 0, g_1^{(2k_0)} \right), & k_0 = 1, \\ G \left(-2\psi_{xyxy}^{(2k_0-2)} - \psi_{xxxx}^{(2k_0-4)}, 0, g_1^{(2k_0)} \right), & k_0 \geq 2. \end{cases}$$

We will use the second formula for both cases with the understanding that $\psi^{(-2)}$ should be replaced by zero. The first step of Algorithm 3.1 is to compute $\psi_0^{(2k_0)}$. Using the induction hypothesis, we obtain

$$(3.28) \quad \psi_0^{(2k_0)} = \frac{I_2}{I_3} \left(-2\mathbb{V}^4 \alpha_{xyy}^{(2k_0-2)} - \mathbb{V}^4 \alpha_{xxx}^{(2k_0-4)} \right) + \sum_{\ell=0}^{k_0-1} Q^{(2\ell)} \left(-2\mathbb{V}^4 \beta_{xyy}^{(2k_0-2\ell-2)} \right) + \sum_{\ell=0}^{k_0-2} Q^{(2\ell)} \left(-\mathbb{V}^4 \beta_{xxx}^{(2k_0-2\ell-4)} \right).$$

The upper limit of the last sum can be replaced by $k_0 - 1$, since we interpret $\beta^{(-2)}$ as zero. We would like to rewrite this in the form

$$(3.29) \quad \psi_0^{(2k_0)} = \frac{I_2}{I_3} \left(\sum_{n=4}^{2k_0+3} \alpha_n^{(2k_0)}(x) \frac{y^n}{h^n} \right) + \sum_{\ell=0}^{k_0-1} Q^{(2\ell)} \left(\sum_{n=4}^{2k_0-2\ell+3} \beta_n^{(2k_0-2\ell)}(x) \frac{y^n}{h^n} \right).$$

If we use the induction hypothesis and substitute (3.20) into (3.28), the operator $\mathbb{V}^4 \partial_y^2$ annihilates a single power of y and antidifferentiates higher powers of y twice. Similarly, \mathbb{V}^4 antidifferentiates all powers of y four times. Thus, for $k = k_0$ and $4 \leq n \leq 2k + 3$, we should define

$$(3.30) \quad \alpha_n^{(2k)}(x) = \frac{-2h^n \partial_x^2 \left(\alpha_{n-2}^{(2k-2)} h^{-n+2} \right)}{n(n-1)} + \frac{-h^n \partial_x^4 \left(\alpha_{n-4}^{(2k-4)} h^{-n+4} \right)}{n(n-1)(n-2)(n-3)}$$

with an identical formula for $\beta_n^{(2k)}$ in terms of $\beta_{n-2}^{(2k-2)}$ and $\beta_{n-4}^{(2k-4)}$. The second term should be omitted when $k = 1$ or $n = 4$, and is zero when $n = 5$. As part of the induction hypothesis, we may assume that (3.30) and its β version hold for $1 \leq k < k_0$ as well, so that each term in the sum over ℓ in (3.28) also has the form described in (3.29). Note that for $n \geq 0$ and any differentiable function $\varphi(x)$,

$$(3.31) \quad \partial_x(h^{-n}\varphi) = h^{-(n+1)}(h\partial_x - nh_x)\varphi.$$

By Lemmas 3.6 and 3.7 below, $h\partial_x$ and multiplication by h_x both map \mathcal{H}_k to \mathcal{H}_{k+1} for all $k \geq 0$. Thus

$$(3.32) \quad \begin{aligned} h^n \partial_x^2 \left(\alpha_{n-2}^{(2k_0-2)} h^{-n+2} \right) &= h[h\partial_x - (n-2)h_x][h\partial_x - (n-3)h_x] \left(h^{-1} \alpha_{n-2}^{(2k_0-2)} \right) \in \frac{I_3}{I_2} \mathcal{V}_1 h \mathcal{H}_{2k_0}, \\ h^n \partial_x^2 \left(\beta_{n-2}^{(2k_0-2)} h^{-n+2} \right) &= [h\partial_x - (n-1)h_x][h\partial_x - (n-2)h_x] \left(\beta_{n-2}^{(2k_0-2)} \right) \in \mathcal{H}_{2k_0}, \end{aligned}$$

with similar formulas for $h^n \partial_x^4 (\alpha_{n-4}^{(2k_0-4)} h^{-n+4})$ and $h^n \partial_x^4 (\beta_{n-4}^{(2k_0-4)} h^{-n+4})$. We conclude that $\alpha_n^{(2k)}$ and $\beta_n^{(2k)}$ have the form claimed in (3.21) when $k = k_0$ and $4 \leq n \leq 2k_0 + 3$. Finally, we obtain $Q^{(2k_0)}$ and $\psi^{(2k_0)}$ from $\psi_0^{(2k_0)}$ in (3.29) using Algorithm 3.1.

They satisfy (3.19) and (3.20) if we set $k = k_0$ and define $\alpha_1^{(2k)} = 0$, $\beta_1^{(2k)} = 0$,

$$\begin{aligned}
 \alpha_2^{(2k)}(x) &= \sum_{n=4}^{2k+3} (n-3)\alpha_n^{(2k)}(x) - V_1 \binom{-1/2}{k} h_x^{2k}, \\
 \beta_2^{(2k)}(x) &= \sum_{n=4}^{2k+3} (n-3)\beta_n^{(2k)}(x), \\
 \alpha_3^{(2k)}(x) &= \sum_{n=4}^{2k+3} (2-n)\alpha_n^{(2k)}(x) + V_1 \binom{-1/2}{k} h_x^{2k}, \\
 \beta_3^{(2k)}(x) &= \sum_{n=4}^{2k+3} (2-n)\beta_n^{(2k)}(x),
 \end{aligned}
 \tag{3.33}$$

$a^{(2k)} = \frac{1}{2I_3} \int_0^1 \frac{\alpha_3^{(2k)}(x)}{h(x)^3} dx$, and $b^{(2k)} = \frac{1}{2I_3} \int_0^1 \frac{\beta_3^{(2k)}(x)}{h(x)^3} dx$. As part of the induction hypothesis, we may assume (3.33) also holds for $1 \leq k < k_0$. The factors of n in (3.33) are due to the terms $\pm\psi_{0,y}(x, h)h$ in the formula for $\psi^{(2k_0)}$ in Algorithm 3.1. The terms $3Q^{(2k_0)}\frac{y^2}{h^2}$ and $-2Q^{(2k_0)}\frac{y^3}{h^3}$ in the formula for $\psi^{(2k_0)}$ are accounted for in (3.19) by extending the upper limit of the sum over ℓ from $k_0 - 1$ to k_0 and noting that $\beta^{(0)}(x, y) = 3\frac{y^2}{h^2} - 2\frac{y^3}{h^3}$. Thus, $\psi^{(2k_0)}$ and $Q^{(2k_0)}$ have the desired form, and $\alpha_n^{(2k_0)}$, $\beta_n^{(2k_0)}$ belong to the appropriate spaces, as claimed. \square

To complete this proof, we need two simple lemmas about the spaces \mathcal{H}_k (which also serve as the foundation for our numerical algorithm described in Appendix A).

LEMMA 3.6. *If $k \geq 0$ and $\varphi \in \mathcal{H}_k$, then $h_x\varphi \in \mathcal{H}_{k+1}$.*

Proof. This follows easily from the definition of \mathcal{H}_k in (3.13). \square

LEMMA 3.7. *If $k \geq 0$ and $\varphi \in \mathcal{H}_k$, then $h\partial_x\varphi \in \mathcal{H}_{k+1}$.*

Proof. If $k = 0$, then $h\partial_x\varphi = 0 \in \mathcal{H}_{k+1}$. Suppose $k_0 \geq 1$, and the result holds for $k < k_0$. Let $\varphi \in \mathcal{H}_{k_0}$ be a monomial. Then there is a $k \in \{1, \dots, k_0\}$ and a monomial $\beta \in \mathcal{H}_{k_0-k}$ such that $\varphi = (h^{k-1}\partial_x^k h)\beta$. But then

$$(3.34) \quad h\partial_x\varphi = (k-1)h_x\varphi + (h^k\partial_x^{k+1}h)\beta + (h^{k-1}\partial_x^k h)(h\partial_x\beta).$$

Evidently, all three terms belong to \mathcal{H}_{k_0+1} , the third due to the induction hypothesis. This result can now be applied term by term for any polynomial $\varphi \in \mathcal{H}_k$. \square

4. Error analysis. To estimate the error of the expansion of ψ and Q through order $2k$, we show that the truncation error satisfies a weak form of the rescaled biharmonic equation (2.14) with data (\mathbf{F}, g_0, g_1) of order ε^{2k+2} . We also prove a uniform coercivity result for the family of bilinear forms involved in the weak formulation, which allows us to bound the truncation error in terms of the data.

Throughout this section, we will treat Ω and $T = [0, 1]_p$ as C^∞ manifolds by identifying the points

$$\begin{aligned}
 (4.1) \quad \Omega : \quad & (0, y) \sim (1, y), \quad 0 < y < h(0), \\
 T : \quad & 0 \sim 1
 \end{aligned}$$

and adding a coordinate chart to each that “wraps around.” In particular: a function in $C^k(\Omega)$ or $C^k(T)$ is understood to have k continuous periodic derivatives; $\partial\Omega = \Gamma_0 \cup \Gamma_1$; $\partial T = \emptyset$; the support of a function $\phi \in C_c^k(\Omega)$ vanishes near Γ_0 and Γ_1 but not necessarily at $x = 0$ and $x = 1$; and the Sobolev spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ are the completions of $C^k(\overline{\Omega})$ and $C_c^k(\Omega)$ in the $\|\cdot\|_k$ norm and thus contain only x -periodic functions with appropriate smoothness at $x = 0, 1$.

4.1. Weak formulation of the rescaled biharmonic equation. An interesting difference between the biharmonic equation and the Poisson equation is that the boundary conditions in the latter are completely specified in the problem statement, whereas one of them (the flux Q) in the former problem must be determined as part of the solution. The integral condition (2.16), which uniquely determines Q , must also be reformulated weakly, since it involves more than two derivatives of ψ . This can be done [14] by slightly enlarging the space of test functions to include functions that are constant along Γ_1 (rather than equal to 0 there). To this end, we define

$$(4.2) \quad \Psi = \left\{ \phi \in H^2(\Omega) : (\phi, \partial_y \phi)|_{\Gamma_0} = (0, 0), (\phi, \partial_y \phi)|_{\Gamma_1} = (\text{const}, 0) \right\}.$$

For ϕ, ψ in $H^2(\Omega)$, we define the bilinear form

$$(4.3) \quad \begin{aligned} a_\varepsilon(\psi, \phi) &= \int_{\Omega} \psi_{yy} \phi_{yy} + 2\varepsilon^2 \psi_{xy} \phi_{xy} + \varepsilon^4 \psi_{xx} \phi_{xx} dA \\ &= a^{(0)}(\psi, \phi) + \varepsilon^2 a^{(2)}(\psi, \phi) + \varepsilon^4 a^{(4)}(\psi, \phi). \end{aligned}$$

To obtain estimates that hold uniformly in ε , it will be useful to work with the weighted norms and seminorms

$$(4.4) \quad \begin{aligned} \|\psi\|_0^2 &= \int_{\Omega} \psi^2 dA, \quad \|\psi\|_{1,\varepsilon}^2 = \int_{\Omega} \psi_y^2 + (\varepsilon \psi_x)^2 dA, \quad \|\psi\|_{2,\varepsilon}^2 = a_\varepsilon(\psi, \psi), \\ \|\psi\|_{1,\varepsilon} &= \sqrt{\|\psi\|_0^2 + \|\psi\|_{1,\varepsilon}^2}, \quad \|\psi\|_{2,\varepsilon} = \sqrt{\|\psi\|_0^2 + \|\psi\|_{1,\varepsilon}^2 + \|\psi\|_{2,\varepsilon}^2}. \end{aligned}$$

For fixed ε , these norms are equivalent to the usual Sobolev norms in which ε is set to 1 in these expressions. We use x to parametrize functions defined on Γ_0 or Γ_1 and define the weighted boundary norm

$$(4.5) \quad \|g\|_{1/2,\varepsilon}^2 = \sum_{k=-\infty}^{\infty} [1 + (2\pi k\varepsilon)^2]^{1/2} |c_k|^2, \quad c_k = \int_0^1 g(x) e^{-2\pi i k x} dx.$$

We equip the dual spaces Ψ' and $H^{-1}(\Omega)^2 = [H_0^1(\Omega)^2]'$ with the weighted norms

$$(4.6) \quad \|l\|_{-2,\varepsilon} = \sup_{\|\psi\|_{2,\varepsilon}=1} |\langle l, \psi \rangle|, \quad \|\mathbf{F}\|_{-1,\varepsilon} = \sup_{\|u\|_{1,\varepsilon}^2 + \|\varepsilon v\|_{1,\varepsilon}^2 = 1} |\langle \mathbf{F}, (u, v) \rangle|.$$

Since $\|\psi\|_{2,\varepsilon}^2 \geq \|\psi_y\|_{1,\varepsilon}^2 + \|\varepsilon \psi_x\|_{1,\varepsilon}^2$, the linear functional $\langle l, \psi \rangle = \langle \mathbf{F}, \nabla \times \psi \rangle$ on Ψ satisfies $\|l\|_{-2,\varepsilon} \leq \|\mathbf{F}\|_{-1,\varepsilon}$.

DEFINITION 4.1 (weak solutions). *Suppose*

$$(4.7) \quad h \in C^{1,1}(T), \quad \mathbf{F} \in H^{-1}(\Omega)^2, \quad g_0 \in H^{1/2}(\Gamma_0), \quad g_1 \in H^{1/2}(\Gamma_1).$$

We say that $(\psi, Q) \in H^2(\Omega) \times \mathbb{R}$ is a weak solution of (2.14)–(2.16) if

$$(4.8) \quad a_\varepsilon(\psi, \phi) = \langle \mathbf{F}, \nabla \times \phi \rangle$$

for all $\phi \in \Psi$ and the boundary conditions

$$(4.9) \quad B\psi = (0, g_0, Q, g_1)$$

hold in the trace sense, where $B\psi := (\psi|_{\Gamma_0}, \psi_y|_{\Gamma_0}, \psi|_{\Gamma_1}, \psi_y|_{\Gamma_1})$.

PROPOSITION 4.2. *Every classical solution is a weak solution.*

Proof. We assume $\psi \in C^4(\overline{\Omega})$ and (2.14)–(2.16) hold classically; this requires additional regularity for \mathbf{F} , g_0 , g_1 , of course. If we multiply (2.14) by a test function $\phi \in C^2(\overline{\Omega}) \cap \Psi$ and use the identity $\chi(\nabla \times \mathbf{v}) = \nabla \times (\chi \mathbf{v}) + (\nabla \times \chi) \cdot \mathbf{v}$, we obtain

$$\begin{aligned}
 (4.10) \quad 0 &= \int_{\Omega} \phi(-\Delta_{\varepsilon}^2 \psi + \nabla \times \mathbf{F}) dA \\
 &= \int_{\Omega} (\nabla \times [\phi(\Delta_{\varepsilon} \mathbf{u}_{\varepsilon} + \mathbf{F})] + (\nabla \times \phi) \cdot [\Delta_{\varepsilon} \mathbf{u}_{\varepsilon} + \mathbf{F}]) dA \\
 &= \int_{\Gamma_0 - \Gamma_1} \phi(\Delta_{\varepsilon} \mathbf{u}_{\varepsilon} + \mathbf{F}) \cdot \mathbf{t} ds + \int_{\Omega} [(\nabla \times \phi)_{\varepsilon} \cdot (\nabla \times \Delta_{\varepsilon} \psi) + (\nabla \times \phi) \cdot \mathbf{F}] dA \\
 &= \int_{\Gamma_0 - \Gamma_1} [\phi(\cdots) - (\Delta_{\varepsilon} \psi)(\nabla \times \phi)_{\varepsilon}] \cdot \mathbf{t} ds + \int_{\Omega} [-(\Delta_{\varepsilon} \phi)(\Delta_{\varepsilon} \psi) + (\nabla \times \phi) \cdot \mathbf{F}] dA,
 \end{aligned}$$

where $(\nabla \times \phi)_{\varepsilon} = (\phi_y, -\varepsilon^2 \phi_x)$ and the curves Γ_0 and Γ_1 are both oriented from left to right as in Figure 2.1. The conditions

$$\begin{aligned}
 (4.11) \quad &\phi|_{\Gamma_0} = 0, & \partial_y \phi|_{\Gamma_0} &= 0, \\
 &\phi|_{\Gamma_1} = \text{const}, & \partial_y \phi|_{\Gamma_1} &= 0,
 \end{aligned}$$

ensure that the boundary terms are zero: the first boundary term is equal to

$$(4.12) \quad (\phi|_{\Gamma_1})[p(1, h(1)) - p(0, h(0))] = 0$$

(with p as in (2.17), where it was shown to be periodic), and the second is zero since $\nabla \times \phi = 0$ on Γ_0 and Γ_1 . One more integration by parts gives $\int_{\Omega} (\Delta_{\varepsilon} \phi)(\Delta_{\varepsilon} \psi) dA = a_{\varepsilon}(\psi, \phi)$, so (4.8) holds. Since $C^2(\overline{\Omega}) \cap \Psi$ is dense in Ψ and both sides of (4.8) are bounded linear functionals of $\phi \in \Psi$, this formula holds for all $\phi \in \Psi$. \square

4.2. Uniform coercivity. The following two theorems are the key to obtaining error estimates for the expansion solutions of section 3.

THEOREM 4.3. *The bilinear form $a_{\varepsilon}(\cdot, \cdot)$ is coercive on Ψ (uniformly in ε) with respect to the weighted norm $\|\cdot\|_{2,\varepsilon}$, i.e., there is a constant $\alpha > 0$ such that $\alpha \|\psi\|_{2,\varepsilon}^2 \leq a_{\varepsilon}(\psi, \psi)$ for all $\varepsilon > 0$, $\psi \in \Psi$.*

Proof. Without loss of generality, we may assume the characteristic height \bar{H} of the domain was chosen so that $0 < h(x) \leq 1$ for $0 \leq x \leq 1$. We now use a standard Poincaré–Friedrichs argument [5]. Suppose $\psi \in C^2(\overline{\Omega}) \cap \Psi$. Then

$$(4.13) \quad |\psi(x, y)|^2 = \left| \int_0^y \psi_y(x, \eta) d\eta \right|^2 \leq y \int_0^{h(x)} |\psi_y(x, \eta)|^2 d\eta,$$

$$(4.14) \quad |\psi(x, y)|^2 = \left| \int_0^y (y - \eta) \psi_{yy}(x, \eta) d\eta \right|^2 \leq \frac{y^3}{3} \int_0^{h(x)} |\psi_{yy}(x, \eta)|^2 d\eta.$$

Integrating over Ω ,

$$(4.15) \quad \|\psi\|_0^2 \leq \frac{h_{\max}^2}{2} \int_{\Omega} |\psi_y|^2 dA \leq \frac{1}{2} |\psi|_{1,\varepsilon}^2, \quad \|\psi\|_0^2 \leq \frac{h_{\max}^4}{12} \int_{\Omega} |\psi_{yy}|^2 dA \leq \frac{1}{12} |\psi|_{2,\varepsilon}^2.$$

Repeating this argument on the derivatives of ψ yields

$$(4.16) \quad |\psi|_{1,\varepsilon}^2 = \|\psi_y\|_0^2 + \|\varepsilon \psi_x\|_0^2 \leq \frac{1}{2} (|\psi_y|_{1,\varepsilon}^2 + |\varepsilon \psi_x|_{1,\varepsilon}^2) = \frac{1}{2} |\psi|_{2,\varepsilon}^2 = \frac{1}{2} a_{\varepsilon}(\psi, \psi)$$

so that $\|\psi\|_{2,\varepsilon}^2 \leq \frac{19}{12}a_\varepsilon(\psi, \psi)$. Since $C^2(\overline{\Omega}) \cap \Psi$ is dense in Ψ , we conclude that $(12/19)\|\psi\|_{2,\varepsilon}^2 \leq a_\varepsilon(\psi, \psi)$ for all $\psi \in \Psi$ as claimed. \square

THEOREM 4.4. *A weak solution ψ of the boundary value problem (2.14)–(2.16) exists and is unique. Moreover, the following estimate holds:*

(4.17)

$$\|\psi\|_{2,\varepsilon} \leq \frac{19}{12}\|\mathbf{F}\|_{-1,\varepsilon} + \left(72 + 860 \left(\varepsilon^2 \|h_x\|_\infty^2 + \varepsilon^4 \|h_x\|_\infty^4 + 4\varepsilon^4 \left\| \frac{1}{2} h h_{xx} \right\|_\infty^2 \right) \right)^{1/2} \left(\|h^{-1/2} g_0\|_{1/2,\varepsilon} + \|h^{-1/2} g_1\|_{1/2,\varepsilon} \right).$$

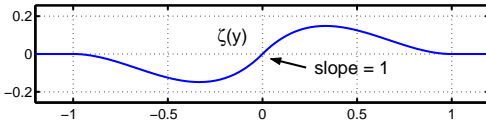
In particular, if $\varepsilon \leq \frac{r_0}{3}$ with $r_0^{-1} = \max(\|h_x\|_\infty, \|\frac{1}{2} h h_{xx}\|_\infty^{1/2})$, then

$$(4.18) \quad \|\psi\|_{2,\varepsilon} \leq \frac{19}{12}\|\mathbf{F}\|_{-1,\varepsilon} + 15 \left(\|h^{-1/2} g_0\|_{1/2,\varepsilon} + \|h^{-1/2} g_1\|_{1/2,\varepsilon} \right).$$

Proof. We begin by constructing a function $\psi_0 \in H^2(\Omega)$ that satisfies the boundary conditions (2.15) with $Q = 0$. First, we map the domain Ω to the x -periodic unit square $R = T \times (0, 1)$ via the transformation

$$(4.19) \quad \tilde{\psi}_0(x, y) = h(x)^{-3/2} \psi_0(x, h(x)y), \quad 0 \leq x \leq 1, \quad 0 < y < 1.$$

We include $h^{-3/2}$ here to avoid powers of h_0^{-1} in (4.27), where $h_0 = \min_{0 \leq x \leq 1} h(x)$. We require $\tilde{\psi}_0(x, 0) = 0$, $\tilde{\psi}_{0,y}(x, 0) = h(x)^{-1/2} g_0(x)$, $\tilde{\psi}_0(x, 1) = 0$, and $\tilde{\psi}_{0,y}(x, 1) = h(x)^{-1/2} g_1(x)$. To construct such a function, we define $\zeta \in C^1(\mathbb{R})$ via

$$\zeta(y) = \begin{cases} 0, & y \leq -1 \\ y + 2y^2 + y^3, & -1 \leq y \leq 0 \\ y - 2y^2 + y^3, & 0 \leq y \leq 1 \\ 0, & 1 \leq y \end{cases} \quad \text{and} \quad \text{graph of } \zeta(y) \text{ on } [-1, 1] \text{ with slope } = 1 \text{ at } y=0.$$


and set

$$(4.20) \quad \tilde{\psi}_0(x, y) = \sum_{k=-\infty}^{\infty} \left(c_k \frac{\zeta(\langle k \rangle y)}{\langle k \rangle} + d_k \frac{\zeta(\langle k \rangle (y-1))}{\langle k \rangle} \right) e^{2\pi i k x},$$

where

$$(4.21) \quad \langle k \rangle = [1 + (2\pi k \varepsilon)^2]^{1/2}, \quad [c_k, d_k] = \int_0^1 [g_0, g_1](x) h(x)^{-1/2} e^{-2\pi i k x} dx.$$

The value and slope of ζ at $y = 0$ and $|y| \geq 1$ ensure that $\tilde{\psi}_0$ satisfies the desired boundary conditions. Assume for the moment that each d_k is zero (i.e., $g_1 \equiv 0$). Let S be the strip $T \times \mathbb{R}$. We may use (4.20) to define $\tilde{\psi}_0$ on all of S and take its Fourier transform

$$(4.22) \quad (\tilde{\psi}_0)^\wedge(k, \eta) = \int_0^1 \int_{-\infty}^{\infty} \tilde{\psi}_0(x, y) e^{-2\pi i(kx + \eta y)} dy dx = \frac{c_k \hat{\zeta}(\eta/\langle k \rangle)}{\langle k \rangle^2}.$$

Since ζ is antisymmetric and supported on $[-1, 1]$, we have

$$\begin{aligned}
 2 \left\| \tilde{\psi}_0 \right\|_{2,\varepsilon,R}^2 &= \left\| \tilde{\psi}_0 \right\|_{2,\varepsilon,S}^2 \leq \left\| \tilde{\psi}_0 \right\|_{0,S}^2 + 2 \left| \tilde{\psi}_0 \right|_{1,\varepsilon,S}^2 + \left| \tilde{\psi}_0 \right|_{2,\varepsilon,S}^2 \\
 (4.23) \quad &= \sum_k \int_{-\infty}^{\infty} [1 + (2\pi k\varepsilon)^2 + (2\pi\eta)^2]^2 \left| \left(\tilde{\psi}_0 \right)^\wedge(k, \eta) \right|^2 d\eta \\
 &= \sum_k \langle k \rangle |c_k|^2 \int_{-\infty}^{\infty} [1 + (2\pi t)^2]^2 \left| \hat{\zeta}(t) \right|^2 dt = \frac{898}{105} \left\| h^{-1/2} g_0 \right\|_{1/2,\varepsilon}^2.
 \end{aligned}$$

A similar argument works if we assume $g_0 \equiv 0$, but $g_1 \neq 0$. Thus, on R , we have

$$(4.24) \quad \left\| \tilde{\psi}_0 \right\|_{2,\varepsilon} \leq \sqrt{\frac{449}{105}} \left(\left\| h^{-1/2} g_0 \right\|_{1/2,\varepsilon} + \left\| h^{-1/2} g_1 \right\|_{1/2,\varepsilon} \right).$$

Next we use the formula $\psi_0(x, y) = h(x)^{3/2} \tilde{\psi}_0(x, \frac{y}{h(x)})$ to obtain

$$\begin{aligned}
 \psi_{0,y} &= h^{1/2} \tilde{\psi}_{0,y}, & \psi_{0,yy} &= h^{-1/2} \tilde{\psi}_{0,yy}, \\
 \psi_{0,x} &= h^{3/2} \tilde{\psi}_{0,x} - y h^{-1/2} h_x \tilde{\psi}_{0,y} + \frac{3}{2} h^{1/2} h_x \tilde{\psi}_0, \\
 (4.25) \quad \psi_{0,xy} &= h^{1/2} \tilde{\psi}_{0,xy} - y h^{-3/2} h_x \tilde{\psi}_{0,yy} + \frac{1}{2} h^{-1/2} h_x \tilde{\psi}_{0,y}, \\
 \psi_{0,xx} &= h^{3/2} \tilde{\psi}_{0,xx} + 3 h^{1/2} h_x \tilde{\psi}_{0,x} - 2 y h^{-1/2} h_x \tilde{\psi}_{0,xy} + \frac{3}{2} h^{1/2} h_{xx} \tilde{\psi}_0 \\
 &\quad + y^2 h^{-5/2} h_x^2 \tilde{\psi}_{0,yy} - y h^{-3/2} h_x^2 \tilde{\psi}_{0,y} - y h^{-1/2} h_{xx} \tilde{\psi}_{0,y} + \frac{3}{4} h^{-1/2} h_x^2 \tilde{\psi}_0.
 \end{aligned}$$

Using Lemma 4.5 below and $0 < h(x) \leq 1$, we find that

$$\begin{aligned}
 (4.26) \quad \int_{\Omega} \psi_0^2 dA &= \int_0^1 \int_0^1 \psi_0(x, h(x)y)^2 h(x) dx dy = \int_R h^4 \tilde{\psi}_0^2 dA \leq \int_R \tilde{\psi}_0^2 dA, \\
 \int_{\Omega} \psi_{0,y}^2 dA &\leq \int_R \tilde{\psi}_{0,y}^2 dA, & \int_{\Omega} \psi_{0,yy}^2 dA &= \int_R \tilde{\psi}_{0,yy}^2 dA, \\
 \int_{\Omega} \varepsilon^2 \psi_{0,x}^2 dA &\leq 2 \int_R \varepsilon^2 \tilde{\psi}_{0,x}^2 dA + 4\varepsilon^2 \|h_x^2\|_{\infty} \left[\int_R \tilde{\psi}_{0,y}^2 dA + \frac{9}{4} \int_R \tilde{\psi}_0^2 dA \right], \\
 \int_{\Omega} 2\varepsilon^2 \psi_{0,xy}^2 dA &\leq 2 \int_R 2\varepsilon^2 \tilde{\psi}_{0,xy}^2 dA + 4\varepsilon^2 \|h_x^2\|_{\infty} \left[2 \int_R \tilde{\psi}_{0,yy}^2 dA + \frac{1}{2} \int_R \tilde{\psi}_{0,y}^2 dA \right], \\
 \int_{\Omega} \varepsilon^4 \psi_{0,xx}^2 dA &\leq \frac{5}{2} \int_R \varepsilon^4 \tilde{\psi}_{0,xx}^2 dA + 30\varepsilon^2 \|h_x^2\|_{\infty} \int_R \varepsilon^2 \tilde{\psi}_{0,x}^2 dA \\
 &\quad + 30\varepsilon^2 \|h_x^2\|_{\infty} \int_R 2\varepsilon^2 \tilde{\psi}_{0,xy}^2 dA + 30\varepsilon^4 \|h^2 h_{xx}^2\|_{\infty} \int_R \tilde{\psi}_0^2 dA \\
 &\quad + 30\varepsilon^4 \|h_x^4\|_{\infty} \int_R \tilde{\psi}_{0,yy}^2 dA + 30\varepsilon^4 \|h_x^4\|_{\infty} \int_R \tilde{\psi}_{0,y}^2 dA \\
 &\quad + 30\varepsilon^4 \|h^2 h_{xx}^2\|_{\infty} \int_R \tilde{\psi}_{0,y}^2 dA + 30\varepsilon^4 \|h_x^4\|_{\infty} \int_R \tilde{\psi}_0^2 dA.
 \end{aligned}$$

Note that the inverse powers of h in (4.25) are canceled when we change variables from Ω to R by the factors of h that arise from the substitution $y \rightarrow hy$ and from the

Jacobian of the transformation. Collecting terms and majorizing, we obtain

$$(4.27) \quad \|\psi_0\|_{2,\varepsilon} \leq (5/2 + 30\varepsilon^2\|h_x\|_\infty^2 + 30\varepsilon^4\|h_x\|_\infty^4 + 30\varepsilon^4\|hh_{xx}\|_\infty^2)^{1/2} \|\tilde{\psi}_0\|_{2,\varepsilon}.$$

Finally, we correct ψ_0 by a function in Ψ to obtain the weak solution ψ , which must satisfy

$$(4.28) \quad \psi - \psi_0 \in \Psi, \quad a_\varepsilon(\psi - \psi_0, \phi) = \langle l, \phi \rangle := \langle \mathbf{F}, \nabla \times \phi \rangle - a_\varepsilon(\psi_0, \phi) \quad \text{for all } \phi \in \Psi.$$

Since l is a bounded linear functional on Ψ , the Lax–Milgram theorem implies existence and uniqueness of the solution ψ of (4.28) and gives the error bound

$$(4.29) \quad \|\psi - \psi_0\|_{2,\varepsilon} \leq \alpha^{-1} \|l\|_{2,\varepsilon} \leq \frac{19}{12} (\|\mathbf{F}\|_{-1,\varepsilon} + \|\psi_0\|_{2,\varepsilon}).$$

Combining this with (4.24) and (4.27) and using the triangle inequality gives (4.17), where we note that $\frac{5}{2}(\frac{31}{12})^2(\frac{449}{105}) \leq 72$ and $30(\frac{31}{12})^2(\frac{449}{105}) \leq 860$. \square

The following lemma was used to balance the coefficients in the terms of (4.25) as much as possible.

LEMMA 4.5. *For any $a_1, \dots, a_8 \in \mathbb{R}$,*

(4.30)

$$\begin{aligned} (a_1 + a_2 + a_3)^2 &\leq 2a_1^2 + 4a_2^2 + 4a_3^2, \\ (a_1 + \dots + a_8)^2 &\leq \frac{5}{2}a_1^2 + \frac{10}{3}a_2^2 + 15a_3^2 + \frac{40}{3}a_4^2 + 30(a_5^2 + a_6^2 + a_7^2) + \frac{160}{3}a_8^2. \end{aligned}$$

Proof. In general, given positive real numbers $\gamma_1, \dots, \gamma_n$ such that $\sum_1^n \gamma_j^{-1} \leq 1$, then for all $a \in \mathbb{R}^n$, we have $(\sum_1^n a_j)^2 \leq \sum_1^n \gamma_j a_j^2$. This is a consequence of the Cauchy–Schwarz inequality:

$$(4.31) \quad \left(\sum_j a_j \right)^2 = \left(\sum_j (\gamma_j^{-1/2}) (\gamma_j^{1/2} a_j) \right)^2 \leq \left(\sum_j \gamma_j^{-1} \right) \left(\sum_j \gamma_j a_j^2 \right).$$

One readily checks that $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$ and $(\frac{2}{5} + \dots + \frac{3}{160}) = \frac{461}{480} \leq 1$. \square

4.3. Truncation error equation. In section 3, we showed how to construct successive terms in the stream function expansion by solving the recursion (3.2)–(3.5). Theorem 3.3 guarantees that derivatives of h higher than $2k$ do not appear in the formulas for $\psi^{(0)}, \dots, \psi^{(2k)}$; hence, if $h \in C^{2k}(T)$, these functions satisfy (3.2)–(3.5) in the classical sense (with k replaced by ℓ and running from 0 to k instead of 0 to ∞). Thus, if $h \in C^{2k+4}$, $\psi_{approx}^{(2k)} = \psi^{(0)} + \varepsilon^2 \psi^{(2)} + \dots + \varepsilon^{2k} \psi^{(2k)}$ satisfies

$$(4.32) \quad \Delta_\varepsilon^2 \psi_{approx}^{(2k)} = \varepsilon^{2k+2} \left(2\psi_{xyy}^{(2k)} + \psi_{xxxx}^{(2k-2)} \right) + \varepsilon^{2k+4} \psi_{xxxx}^{(2k)}.$$

The truncation error $\psi_{err}^{(2k)} = \psi_{exact} - \psi_{approx}^{(2k)}$ then satisfies $\Delta_\varepsilon^2 \psi_{err} = -\Delta_\varepsilon^2 \psi_{approx}$ with $O(\varepsilon^{2k+2})$ boundary data. Since the right-hand side of (4.32) and the boundary data are known in terms of h , we are able to estimate the size of $\psi_{err}^{(2k)}$ using Theorem 4.4 above. However, to use this theorem, we need to formulate (4.32) weakly.

We begin by showing that the $\psi^{(2\ell)}$ satisfy a weak version of the recursion (3.2). Suppose $k \geq 0$ and $h \in C^{2k}(T)$. Let $\phi \in \Psi$ and denote the constant value of ϕ on Γ_1 by q . We multiply both sides of (3.2) by ϕ and integrate by parts using

$$(4.33) \quad \begin{aligned} \int_{\Omega} \phi \psi_{yyyy}^{(2\ell)} dA &= \int_{\Omega} \phi_{yy} \psi_{yy}^{(2\ell)} dA + q \int_0^1 \psi_{yyy}^{(2\ell)}(x, h(x)) dx, \\ 2 \int_{\Omega} \phi \psi_{xxyy}^{(2\ell)} dA &= 2 \int_{\Omega} \phi_{xy} \psi_{xy}^{(2\ell)} dA + q \int_0^1 [\psi_{xxy}^{(2\ell)}(x, h(x)) - \psi_{xyy}^{(2\ell)}(x, h(x)) h_x] dx, \\ \int_{\Omega} \phi \psi_{xxxx}^{(2\ell)} dA &= \int_{\Omega} \phi_{xx} \psi_{xx}^{(2\ell)} dA - q \int_0^1 \psi_{xxx}^{(2\ell)}(x, h(x)) h_x dx \end{aligned}$$

to obtain the recursion

$$(4.34) \quad \begin{aligned} a^{(0)}(\psi^{(0)}, \phi) &= 0, \\ a^{(0)}(\psi^{(2)}, \phi) &= -a^{(2)}(\psi^{(0)}, \phi), \\ a^{(0)}(\psi^{(2\ell)}, \phi) &= -a^{(2)}(\psi^{(2\ell-2)}, \phi) - a^{(4)}(\psi^{(2\ell-4)}, \phi), \quad \ell = 2, 3, \dots, k. \end{aligned}$$

By (3.8), the boundary terms in (4.33) combine to form

$$(4.35) \quad q \int_0^1 [p_x^{(2\ell)}(x, h(x)) + p_y^{(2\ell)}(x, h(x)) h_x] dx = 0, \quad 0 \leq \ell \leq k,$$

when substituted into (3.2). Other boundary terms do not arise in (4.33), since $\phi = 0$ on Γ_0 and $\phi_x = \phi_y = 0$ on Γ_0 and Γ_1 .

Now suppose $h \in C^{2k+1,1}(T)$, i.e., $h(x)$ has $2k+1$ continuous periodic derivatives and $\partial_x^{2k+1} h$ is Lipschitz continuous so that $\partial_x^{2k+2} h \in L^\infty(T)$. Let $(\psi_{exact}, Q_{exact})$ be the weak solution of

$$(4.36) \quad \Delta_\varepsilon^2 \psi = 0, \quad B\psi = (0, g_0, Q, g_1),$$

with g_0, g_1 given in (2.4), and define the truncation errors and approximate solutions

$$(4.37) \quad \begin{aligned} \psi_{err}^{(2k)} &= \psi_{exact} - \psi_{approx}^{(2k)}, & \psi_{approx}^{(2k)} &= \psi^{(0)} + \varepsilon^2 \psi^{(2)} + \dots + \varepsilon^{2k} \psi^{(2k)}, \\ Q_{err}^{(2k)} &= Q_{exact} - Q_{approx}^{(2k)}, & Q_{approx}^{(2k)} &= Q^{(0)} + \varepsilon^2 Q^{(2)} + \dots + \varepsilon^{2k} Q^{(2k)}. \end{aligned}$$

Since $\psi^{(0)}, \dots, \psi^{(2k)}$ satisfy (4.34) and (3.3) while $a_\varepsilon(\psi_{exact}, \phi) = 0$ for every $\phi \in \Psi$, we may expand $a_\varepsilon(\psi_{err}^{(2k)}, \phi)$ in powers of ε to obtain the truncation error equation

$$(4.38) \quad \begin{aligned} a_\varepsilon(\psi_{err}^{(2k)}, \phi) &= -\varepsilon^{2k+2} \langle \mathbf{F}_k, \nabla \times \phi \rangle, & \phi \in \Psi, \\ B\psi_{err}^{(2k)} &= (0, 0, Q_{err}^{(2k)}, \varepsilon^{2k+2} \gamma_k), \end{aligned}$$

where

$$(4.39) \quad \gamma_k = \varepsilon^{-2k-2} \left(g_1 - \left[g_1^{(0)} + \varepsilon^2 g_1^{(2)} + \dots + \varepsilon^{2k} g_1^{(2k)} \right] \right)$$

and

$$\left\langle \mathbf{F}_k, \begin{pmatrix} \phi_y \\ -\phi_x \end{pmatrix} \right\rangle = \begin{cases} a^{(2)}(\psi^{(0)}, \phi) + \varepsilon^2 a^{(4)}(\psi^{(0)}, \phi), & k = 0, \\ a^{(2)}(\psi^{(2k)}, \phi) + \varepsilon^2 a^{(4)}(\psi^{(2k)}, \phi) + a^{(4)}(\psi^{(2k-2)}, \phi), & k \geq 1. \end{cases}$$

There are many functionals $\mathbf{F}_k \in H^{-1}(\Omega)^2$ that have this action on the subspace

$$(4.40) \quad V = \{\nabla \times \phi : \phi \in \Psi\} = \{(u, v) \in H_0^1(\Omega)^2 : u_x + v_y = 0\}.$$

For example, $\mathbf{F}_k = \varepsilon^{-2k-2}[\nabla p_{approx}^{(2k)} - (\Delta_\varepsilon \mathbf{u}_{approx}^{(2k)})_\varepsilon]$ satisfies $\Delta_\varepsilon^2 \psi_{approx}^{(2k)} = \varepsilon^{2k+2} \nabla \times \mathbf{F}_k$ classically and, using (3.8), may be shown to have following action on $H_0^1(\Omega)^2$:

$$(4.41) \quad \begin{aligned} \langle \mathbf{F}_k, (u; v) \rangle &= \int_{\Omega} \left(\psi_{xx}^{(0)} \right) (u_y - \varepsilon^2 v_x) - \left(\varepsilon^{-1} \psi_{xy}^{(0)} \right) (\varepsilon v_y) dA, & k = 0, \\ \langle \mathbf{F}_k, (u; v) \rangle &= \int_{\Omega} \left(\begin{aligned} &\left(\psi_{xx}^{(2k)} \right) (u_y - \varepsilon^2 v_x) - \left(\varepsilon^{-1} \psi_{xy}^{(2k)} \right) (\varepsilon v_y) \\ &- \left(\varepsilon^{-2} \psi_{xx}^{(2k-2)} \right) (\varepsilon^2 v_x) \end{aligned} \right) dA, & k \geq 1. \end{aligned}$$

This choice is suboptimal because the terms $\varepsilon^{-1} \psi_{xy}^{(2k)}$ and $\varepsilon^{-2} \psi_{xx}^{(2k-2)}$ diverge as $\varepsilon \rightarrow 0$. We grouped ε with v_y and ε^2 with v_x due to the definition (4.6) of $\|\mathbf{F}_k\|_{-1, \varepsilon}$. Instead, we will use the following functional, which agrees with (4.41) on V :

$$(4.42) \quad \begin{aligned} \langle \mathbf{F}_k, (u; v) \rangle &= \int_{\Omega} \left(\psi_{xx}^{(0)} \right) (2u_y - \varepsilon^2 v_x) dA, & k = 0, \\ \langle \mathbf{F}_k, (u; v) \rangle &= \int_{\Omega} \left(\psi_{xx}^{(2k)} \right) (2u_y - \varepsilon^2 v_x) + h^2 \psi_{xxxx}^{(2k-2)} \tilde{\phi}[u] dA, & k \geq 1, \\ \tilde{\phi}[u](x, y) &:= - \int_y^{h(x)} \frac{u(x, \eta)}{h(x)^2} d\eta = \int_y^{h(x)} \frac{(\eta - y) u_y(x, \eta)}{h(x)^2} d\eta. \end{aligned}$$

Note that if $\phi \in \Psi$ and $q = \phi|_{\Gamma_1}$, then $\tilde{\phi}[\phi_y] = (\phi - q)h^{-2}$. The purpose of the h^{-2} here is to be able to include an h^2 with $\psi_{xxxx}^{(2k-2)}$ in the error estimates below (to properly consolidate terms). Another alternative to (4.42) that would lead to similar estimates below is $\langle \mathbf{F}_k, (u; v) \rangle = \int_{\Omega} [-\varepsilon^2 \psi_{xx}^{(2k)} v_x - \psi_{yy}^{(2k+2)} u_y] dA$.

4.4. Error estimates. Let us assume from now on that $\varepsilon \leq \frac{r_0}{3}$ with $r_0 = \max(\|h_x\|_{\infty}, \|\frac{1}{2} h h_{xx}\|_{\infty})^{1/2}$. Then by (4.38) and Theorem 4.4, we have

$$(4.43) \quad \left\| \psi_{err}^{(2k)} \right\|_{2, \varepsilon} \leq \varepsilon^{2k+2} \left(\frac{19}{12} \|\mathbf{F}_k\|_{-1, \varepsilon} + 15 \left\| h^{-1/2} \gamma_k \right\|_{1/2, \varepsilon} \right).$$

It remains to bound the norms of \mathbf{F}_k and γ_k in terms of h . From (4.42), we have

$$(4.44) \quad \left| \tilde{\phi}[u](x, y) \right|^2 \leq \left(\int_y^{h(x)} \frac{(\eta - y)^2}{h(x)^4} d\eta \right) \left(\int_0^{h(x)} |u_y(x, \eta)|^2 d\eta \right),$$

where the first integral is $\frac{(h-y)^3}{3h^4}$ and the second is independent of y . Hence

$$(4.45) \quad \left\| \tilde{\phi}[u] \right\|_0^2 = \int_0^1 \int_0^{h(x)} \left| \tilde{\phi}[u](x, y) \right|^2 dy dx \leq \frac{1}{12} \|u_y\|_0^2.$$

From (4.42), we then have

$$(4.46) \quad \begin{aligned} |\langle \mathbf{F}_k, (u; v) \rangle| &\leq \left(2a + \frac{b}{\sqrt{12}} \right) \|u_y\|_0 + a \|\varepsilon^2 v_x\|_0 \\ &\leq \left(5a^2 + \frac{4ab}{\sqrt{12}} + \frac{b^2}{12} \right)^{1/2} (\|u\|_{1, \varepsilon}^2 + \|\varepsilon v\|_{1, \varepsilon}^2)^{1/2}, \end{aligned}$$

where $a = \|\psi_{xx}^{(2k)}\|_0$ and $b = \|h^2\psi_{xxxx}^{(2k-2)}\|_0$. Using $\frac{4ab}{\sqrt{12}} \leq \frac{5}{4}a^2 + \frac{4}{15}b^2$, we find that

$$(4.47) \quad \|\mathbf{F}_k\|_{-1,\varepsilon} \leq \sqrt{\frac{25}{4}a^2 + \frac{7}{20}b^2} \leq \frac{5}{2}a + \frac{3}{5}b, \quad \frac{19}{12}\|\mathbf{F}_k\|_{-1,\varepsilon} \leq 4a + b.$$

Finally, by (4.43), we obtain

$$(4.48) \quad \left\| \psi_{err}^{(2k)} \right\|_{2,\varepsilon} \leq \varepsilon^{2k+2} \left(4 \left\| \psi_{xx}^{(2k)} \right\|_0 + \left\| h^2 \psi_{xxxx}^{(2k-2)} \right\|_0 + 15 \left\| h^{-1/2} \gamma_k \right\|_{1/2,\varepsilon} \right),$$

where the fourth derivative term should be omitted when $k = 0$. In section 4.5, the following bound will also prove useful:

$$(4.49) \quad \left\| \psi_{err}^{(2k)} \right\|_{2,\varepsilon} + \varepsilon^{2k+2} \left\| \psi_{xx}^{(2k)} \right\|_0 \leq \varepsilon^{2k+2} \left(5 \left\| \psi_{xx}^{(2k)} \right\|_0 + \left\| h^2 \psi_{xxxx}^{(2k-2)} \right\|_0 + 15 \left\| \frac{\gamma_k}{h^{1/2}} \right\|_{1/2,\varepsilon} \right).$$

The truncation error in the flux expansion satisfies

$$(4.50) \quad Q_{err}^{(2k)} = \psi_{err}^{(2k)}(x, h(x)) = \int_0^{h(x)} (h(x) - \eta) \frac{\partial^2 \psi_{err}^{(2k)}}{\partial y^2}(x, \eta) d\eta$$

for any $x \in T$. Using estimates similar to (4.44) and (4.45), we find that

$$(4.51) \quad |Q_{err}^{(2k)}| \leq \frac{1}{\sqrt{3}} \left\| \psi_{err}^{(2k)} \right\|_{2,\varepsilon}.$$

Thus, we have reduced the problem of bounding the truncation errors to that of checking the norms of three quantities that can be computed explicitly in closed form.

We begin by attacking the boundary term $\|h^{-1/2}\gamma_k\|_{1/2,\varepsilon}$. This norm was defined in (4.5) above. Recall that

$$(4.52) \quad \gamma_k = V_1 \left([1 + \varepsilon^2 h_x^2]^{-1/2} - \sum_{\ell=0}^k \binom{-1/2}{\ell} (\varepsilon^2 h_x^2)^\ell \right) \varepsilon^{-2k-2},$$

where $\binom{-1/2}{0} = 1$ and for $\ell \geq 1$,

$$(4.53) \quad \binom{-1/2}{\ell} = \frac{(-1/2)(-3/2)\cdots(-[\ell-1/2])}{(1)(2)\cdots(\ell)} = (-1)^\ell \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2\ell-1}{2\ell}.$$

Taking the logarithm of this product and its inverse, one may show that

$$(4.54) \quad \frac{1}{\sqrt{4\ell+1}} \leq \left| \binom{-1/2}{\ell} \right| \leq \frac{1}{\sqrt{3\ell+1}}, \quad \ell = 0, 1, 2, \dots$$

Since $h \in C^{2k+1,1}(T) \subset C^{1,1}(T)$, we know γ_k is at least Lipschitz continuous on T and so belongs to $H^1(T)$. As a result,

$$(4.55) \quad \begin{aligned} \left\| \frac{\gamma_k}{h^{1/2}} \right\|_{1/2,\varepsilon}^2 &\leq \left\| \frac{\gamma_k}{h^{1/2}} \right\|_{1,\varepsilon}^2 = \int_0^1 h^{-1} \gamma_k^2 + \varepsilon^2 \left(-\frac{h^{-3/2}}{2} h_x \gamma_k + h^{-1/2} \gamma_{k,x} \right)^2 dx \\ &\leq \int_0^1 h^{-1} \gamma_k^2 + \frac{5}{4} \varepsilon^2 h^{-3} h_x^2 \gamma_k^2 + \frac{5}{4} \varepsilon^2 h^{-1} \gamma_{k,x}^2 dx. \end{aligned}$$

Since $\varepsilon \|h_x\|_\infty \leq 1/3 < 1$, the binomial expansion of $[1 + \varepsilon^2 h_x^2]^{-1/2}$ converges uniformly on $T = [0, 1]_p$. As the terms in this expansion alternate in sign, the error in truncating the series is smaller (pointwise) than the first omitted term. Therefore, for each $x \in T$,

$$(4.56) \quad |\gamma_k(x)| \leq V_1 \left| \binom{-1/2}{k+1} \right| h_x(x)^{2k+2} \leq \frac{V_1}{\sqrt{3k+4}} h_x(x)^{2k+2}.$$

Since $(-2\ell) \binom{-1/2}{\ell} = \binom{-3/2}{\ell-1}$, by differentiating (4.52) we obtain

$$(4.57) \quad \gamma_{k,x} = V_1 \left([1 + \varepsilon^2 h_x^2]^{-3/2} - \sum_{\ell=0}^{k-1} \binom{-3/2}{\ell} (\varepsilon^2 h_x^2)^\ell \right) (-\varepsilon^2 h_x h_{xx}) \varepsilon^{-2k-2}.$$

The terms in the expansion of $[1 + \varepsilon^2 h_x^2]^{-3/2}$ also alternate in sign, so

$$(4.58) \quad |\gamma_{k,x}(x)| \leq V_1 \left| \binom{-3/2}{k} h_x(x)^{2k+1} h_{xx}(x) \right| \leq V_1 \frac{2(k+1)}{\sqrt{3k+4}} |h_x(x)^{2k+1} h_{xx}(x)|.$$

Combining (4.55), (4.56), and (4.58) and using $\frac{(k+1)^2}{3k+4} \leq \frac{1}{4} + \frac{1}{3}k$, we find that

$$(4.59) \quad \begin{aligned} \left\| \frac{\gamma_k}{h^{1/2}} \right\|_{1/2,\varepsilon}^2 &\leq \frac{V_1^2}{3k+4} \int_0^1 \frac{h_x^{4k+4}}{h} + \frac{5}{4} \frac{\varepsilon^2 h_x^2}{h^3} \left[h_x^{4k+4} + 16(k+1)^2 h_x^{4k} \left(\frac{h h_{xx}}{2} \right)^2 \right] dx \\ &\leq \frac{V_1^2}{4} I_1 \tilde{E}_{1,1}^{(2k+2)} + V_1^2 \varepsilon^2 \|h_x\|_\infty^2 \left[\frac{5}{16} I_3 \tilde{E}_{3,1}^{(2k+2)} + \left(5 + \frac{20}{3}k \right) I_3 \tilde{E}_{3,2}^{(2k+2)} \right] \\ &\leq V_1^2 I_1 \left[\frac{1}{4} + \frac{\varepsilon^2 \|h_x\|_\infty^2}{I_1/I_3} \left(\frac{85}{16} + \frac{20}{3}k \right) \right] \max_{(m,j) \in \{(1,1), (3,1), (3,2)\}} \tilde{E}_{m,j}^{(2k+2)}, \end{aligned}$$

where

$$(4.60) \quad I_m = \int_0^1 \frac{1}{h(x)^m} dx, \quad \tilde{E}_{m,j}^{(2k)} = \frac{1}{I_m} \int_0^1 \frac{(\varphi_j^{(2k)}(x))^2}{h(x)^m} dx.$$

Recall that $\Phi_{2k} = \{\varphi_1^{(2k)}, \dots, \varphi_{d_{2k}}^{(2k)}\} = \{h_x^{2k}, \frac{1}{2}h h_x^{2k-2} h_{xx}, \dots, \frac{1}{(2k)!} h^{2k-1} \partial_x^{2k} h\}$ is a basis for the space \mathcal{H}_{2k} defined in (3.13). $\tilde{E}_{m,j}^{(2k)}$ is the square of the 2-norm (or second moment) of $\varphi_j^{(2k)}(x)$ with respect to the probability measure $I_m^{-1} h(x)^{-m} dx$, whereas $E_{m,j}^{(2k)}$ in (3.24) is the expected value of $\varphi_j^{(2k)}(x)$ with respect to this measure. Using (4.59) in (4.48) gives a bound on the error caused by failing to satisfy the boundary conditions in the stream function expansion. It is perhaps surprising that this bound can be expressed in terms of three simple integrals involving h and its derivatives.

Remark 4.6. I_1 and I_3 are dimensionless quantities in (4.59) — if h and x still carried dimensions of length, an extra length scale (e.g., h_{\max} , which is currently set to 1) would need to be included in the Sobolev norms to allow $\|\psi\|_0^2$, $|\psi|_{1,\varepsilon}^2$, and $|\psi|_{2,\varepsilon}^2$ to be added together; this length scale would also appear in (4.59) to nondimensionalize I_1/I_3 in the denominator.

Finally, we estimate $\|\psi_{xx}^{(2k)}\|_0$ and $\|h^2 \psi_{xxxx}^{(2k-2)}\|_0$ in (4.48) in terms of h . It will be useful in our analysis to define

$$(4.61) \quad r_k = \left(\max_{1 \leq \ell \leq 2k+2} \left\{ \left\| \frac{1}{\ell!} h^{\ell-1} \partial_x^\ell h \right\|_\infty^{1/\ell} \right\} \right)^{-1}$$

so that for $0 \leq \ell \leq k+1$, $1 \leq j \leq d_{2\ell}$, and $m = 1, 2, 3$, we have

$$(4.62) \quad \left| \varphi_j^{(2\ell)}(x) \right| \leq r_k^{-2\ell}, \quad \left| E_{m,j}^{(2\ell)} \right| \leq r_k^{-2\ell}, \quad 0 \leq \tilde{E}_{m,j}^{(2\ell)} \leq r_k^{-4\ell}.$$

If h is real analytic as well as periodic, a standard contour integral argument shows that there is an $r > 0$ such that $\|\partial_x^k h\|_\infty \leq k! r^{-k}$ for all $k \geq 0$. Such an r serves as a common lower bound for r_k in (4.61) for all $k \geq 0$. If h is a constant function, the results below hold with $r_k = \infty$ and $r_k^{-1} = 0$ (i.e., the lubrication approximation is exact).

Our first task will be to bound the growth of the terms $Q^{(2k)}$ in the expansion of the flux. By Theorem 3.3 and Remark 3.4, there are rational matrices $A_0^{(2k)}$, $A_1^{(2k)}$, $B^{(2k)}$ with rows indexed from 0 to $2k+3$ and columns indexed from 1 to d_{2k} such that

$$(4.63) \quad Q^{(2k)} = \frac{I_2}{I_3} a^{(2k)} + \sum_{\ell=0}^{k-1} Q^{(2\ell)} b^{(2k-2\ell)}, \quad k \geq 0,$$

where $a^{(2k)} = \frac{1}{2}[V_0 A_0^{(2k)}(3, \cdot) + V_1 A_1^{(2k)}(3, \cdot)] E_2^{(2k)}$ and $b^{(2k)} = \frac{1}{2} B^{(2k)}(3, \cdot) E_3^{(2k)}$. See Example 3.5 above for a reminder of how this works. We now use (4.62) together with the fact that $|v \cdot w| \leq \|v\|_1 \|w\|_\infty$ for $v, w \in \mathbb{R}^d$ to conclude that

$$(4.64) \quad \left| a^{(2\ell)} \right| \leq \left(|V_0| \kappa_0^{(2\ell)} + |V_1| \kappa_1^{(2\ell)} \right) r_k^{-2\ell}, \quad \left| b^{(2\ell)} \right| \leq \kappa_2^{(2\ell)} r_k^{-2\ell}, \quad 0 \leq \ell \leq k,$$

where

$$(4.65) \quad \kappa_i^{(2k)} = \frac{1}{2} \sum_{j=1}^{d_{2k}} \left| A_i^{(2k)}(3, j) \right|, \quad i = 0, 1, \quad \kappa_2^{(2k)} = \frac{1}{2} \sum_{j=1}^{d_{2k}} \left| B^{(2k)}(3, j) \right|, \quad k \geq 0.$$

It follows from (4.63) that if we increase $\kappa_0^{(2k)}$, $\kappa_1^{(2k)}$ via the loop

for $k = 1, 2, 3, \dots$

$$(4.66) \quad \kappa_i^{(2k)} = \kappa_i^{(2k)} + \sum_{\ell=0}^{k-1} \kappa_i^{(2\ell)} \kappa_2^{(2k-2\ell)}, \quad (i = 0, 1),$$

then

$$(4.67) \quad \left| Q^{(2\ell)} \right| \leq \frac{I_2}{I_3} \left(|V_0| \kappa_0^{(2\ell)} + |V_1| \kappa_1^{(2\ell)} \right) r_k^{-2\ell}, \quad (k \geq 0, 0 \leq \ell \leq k).$$

The constants $\kappa_0^{(2k)}$, $\kappa_1^{(2k)}$, $\kappa_2^{(2k)}$ do not depend on h and may be computed once and for all along with the matrices $A_i^{(2k)}$ and $B^{(2k)}$; see Table 4.1.

Now that the terms $Q^{(2\ell)}$ have been bounded, we are ready to estimate $\|\psi_{xx}^{(2k)}\|_0$ and $\|h^2 \psi_{xxxx}^{(2k-2)}\|_0$. Recall that

$$(4.68) \quad \psi^{(2k)}(x, y) = \frac{I_2}{I_3} \alpha^{(2k)}(x, y) + \sum_{\ell=0}^k Q^{(2\ell)} \beta^{(2k-2\ell)}(x, y),$$

where $\alpha^{(2k)}(x, y)$ and $\beta^{(2k)}(x, y)$ have the matrix representations (3.22). A slight modification of the proof of Theorem 3.3 shows that there are also matrices $\dot{A}_0^{(2k)}$,

TABLE 4.1
 $\kappa_i^{(2k)}$ before and after loop (4.66).

k	$\kappa_0^{(2k)}$ before	$\kappa_1^{(2k)}$ before	$\kappa_2^{(2k)}$	$\kappa_0^{(2k)}$ after	$\kappa_1^{(2k)}$ after
0	5.00×10^{-01}	5.00×10^{-01}	$1.00 \times 10^{+00}$	5.00×10^{-01}	5.00×10^{-01}
1	3.00×10^{-01}	5.83×10^{-01}	8.00×10^{-01}	7.00×10^{-01}	9.83×10^{-01}
2	5.30×10^{-01}	7.05×10^{-01}	$1.73 \times 10^{+00}$	$1.96 \times 10^{+00}$	$2.36 \times 10^{+00}$
3	$2.72 \times 10^{+00}$	$3.73 \times 10^{+00}$	$6.74 \times 10^{+00}$	$8.87 \times 10^{+00}$	$1.07 \times 10^{+01}$
4	$1.83 \times 10^{+01}$	$3.32 \times 10^{+01}$	$4.14 \times 10^{+01}$	$5.43 \times 10^{+01}$	$7.32 \times 10^{+01}$
5	$2.00 \times 10^{+02}$	$3.69 \times 10^{+02}$	$4.55 \times 10^{+02}$	$5.28 \times 10^{+02}$	$7.30 \times 10^{+02}$
6	$3.41 \times 10^{+03}$	$6.32 \times 10^{+03}$	$7.22 \times 10^{+03}$	$8.00 \times 10^{+03}$	$1.13 \times 10^{+04}$
7	$7.77 \times 10^{+04}$	$1.66 \times 10^{+05}$	$1.54 \times 10^{+05}$	$1.68 \times 10^{+05}$	$2.63 \times 10^{+05}$
8	$2.69 \times 10^{+06}$	$5.23 \times 10^{+06}$	$4.69 \times 10^{+06}$	$5.31 \times 10^{+06}$	$7.98 \times 10^{+06}$
9	$1.26 \times 10^{+08}$	$2.31 \times 10^{+08}$	$1.94 \times 10^{+08}$	$2.31 \times 10^{+08}$	$3.40 \times 10^{+08}$
10	$6.51 \times 10^{+09}$	$1.45 \times 10^{+10}$	$9.97 \times 10^{+09}$	$1.18 \times 10^{+10}$	$2.00 \times 10^{+10}$

$\dot{A}_1^{(2k)}$, $\dot{B}^{(2k)}$ and $\ddot{A}_0^{(2k)}$, $\ddot{A}_1^{(2k)}$, $\ddot{B}^{(2k)}$ of dimension $(2k+4) \times d_{2k+2}$ and $(2k+4) \times d_{2k+4}$, respectively, such that

$$\begin{aligned}
 \frac{I_2}{I_3} \alpha_{xx}^{(2k)}(x, y) &= h(x)^{-1} (Y_{2k}(x, y))^T \left[V_0 \dot{A}_0^{(2k)} + V_1 \dot{A}_1^{(2k)} \right] \Phi_{2k+2}(x), \\
 \beta_{xx}^{(2k)}(x, y) &= h(x)^{-2} (Y_{2k}(x, y))^T \dot{B}^{(2k)} \Phi_{2k+2}(x), \\
 \frac{I_2}{I_3} \alpha_{xxx}^{(2k)}(x, y) &= h(x)^{-3} (Y_{2k}(x, y))^T \left[V_0 \ddot{A}_0^{(2k)} + V_1 \ddot{A}_1^{(2k)} \right] \Phi_{2k+4}(x), \\
 (4.69) \quad \beta_{xxx}^{(2k)}(x, y) &= h(x)^{-4} (Y_{2k}(x, y))^T \ddot{B}^{(2k)} \Phi_{2k+4}(x),
 \end{aligned}$$

where $Y_{2k} = (1, \frac{y}{h}, \dots, (\frac{y}{h})^{2k+3})^T$. We can achieve significantly sharper estimates of $\|\psi_{xx}^{(2k)}\|_0$ and $\|h^2 \psi_{xxx}^{(2k-2)}\|_0$ by expressing the dependence of ψ on y using orthogonal polynomials. Let

$$(4.70) \quad \tilde{Y}_{2k} = \left(\tilde{P}_0\left(\frac{y}{h}\right), \tilde{P}_1\left(\frac{y}{h}\right), \dots, \tilde{P}_{2k+3}\left(\frac{y}{h}\right) \right)^T$$

be the vector of shifted Legendre polynomials [1], which satisfy

$$(4.71) \quad \int_0^1 \tilde{P}_m(x) \tilde{P}_n(x) dx = \frac{\delta_{mn}}{2n+1}, \quad m, n \geq 0.$$

The first several are

$$(4.72) \quad \tilde{P}_0 = 1, \quad \tilde{P}_1 = 2x - 1, \quad \tilde{P}_2 = 6x^2 - 6x + 1, \quad \tilde{P}_3 = 20x^3 - 30x^2 + 12x - 1.$$

The well-known recurrence [1]

$$(4.73) \quad \tilde{P}_n(x) = \frac{2n-1}{n} (2x-1) \tilde{P}_{n-1}(x) - \frac{n-1}{n} \tilde{P}_{n-2}(x), \quad n \geq 2,$$

can be used to construct a nested family of lower triangular matrices R_{2k} of dimension $(2k+4) \times (2k+4)$ with indices starting at zero such that

$$(4.74) \quad \tilde{Y}_{2k} = R_{2k} Y_{2k}, \quad Y_{2k}^T = \tilde{Y}_{2k}^T R_{2k}^{-T}.$$

For example,

$$(4.75) \quad R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -6 & 6 & 0 \\ -1 & 12 & -30 & 20 \end{pmatrix}, \quad R_0^{-T} = \begin{pmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 0 & 1/2 & 1/2 & 9/20 \\ 0 & 0 & 1/6 & 1/4 \\ 0 & 0 & 0 & 1/20 \end{pmatrix}.$$

The entries of R_{2k}^{-T} are nonnegative and have unit column sums for all $k \geq 0$. Next we renormalize the shifted Legendre polynomials and define

$$(4.76) \quad \begin{aligned} \dot{P}_n(x, y) &= h(x)^{-1/2} \sqrt{2n+1} \tilde{P}_n(y/h(x)), \\ \dot{Y}_{2k} &= \left(\dot{P}_0, \dots, \dot{P}_{2k+3} \right)^T = h^{-1/2} D_{2k} \tilde{Y}_{2k}, \quad D_{2k} = \text{diag} \left(\sqrt{1}, \sqrt{3}, \dots, \sqrt{4k+7} \right) \end{aligned}$$

so that (4.69) becomes

$$(4.77) \quad \begin{aligned} \frac{I_2}{I_3} \alpha_{xx}^{(2k)} &= \left(\dot{Y}_{2k}(x, y) \right)^T D_{2k}^{-1} R_{2k}^{-T} \left[V_0 \dot{A}_0^{(2k)} + V_1 \dot{A}_1^{(2k)} \right] \left(h(x)^{-1/2} \Phi_{2k+2}(x) \right), \\ \beta_{xx}^{(2k)} &= \left(\dot{Y}_{2k}(x, y) \right)^T D_{2k}^{-1} R_{2k}^{-T} \dot{B}^{(2k)} \left(h(x)^{-3/2} \Phi_{2k+2}(x) \right), \\ \frac{I_2}{I_3} h^2 \alpha_{xxxx}^{(2k)} &= \left(\dot{Y}_{2k}(x, y) \right)^T D_{2k}^{-1} R_{2k}^{-T} \left[V_0 \ddot{A}_0^{(2k)} + V_1 \ddot{A}_1^{(2k)} \right] \left(h(x)^{-1/2} \Phi_{2k+4}(x) \right), \\ h^2 \beta_{xxxx}^{(2k)} &= \left(\dot{Y}_{2k}(x, y) \right)^T D_{2k}^{-1} R_{2k}^{-T} \ddot{B}^{(2k)} \left(h(x)^{-3/2} \Phi_{2k+4}(x) \right). \end{aligned}$$

Example 4.7. From (3.9) and (3.10), we have

$$(4.78) \quad \begin{aligned} Q^{(0)} &= \frac{V_0 + V_1}{2} \frac{I_2}{I_3}, \\ \psi_{xx}^{(0)} &= \left[\left(V_0 + \frac{V_1}{2} \right) \left(\frac{-4h_x^2 + 2hh_{xx}}{h} \right) + Q^{(0)} \left(\frac{18h_x^2 - 6hh_{xx}}{h^2} \right) \right] \frac{y^2}{h^2} \\ &\quad + \left[(V_0 + V_1) \left(\frac{6h_x^2 - 2hh_{xx}}{h} \right) + Q^{(0)} \left(\frac{-24h_x^2 + 6hh_{xx}}{h^2} \right) \right] \frac{y^3}{h^3}, \end{aligned}$$

which has the form described in (4.68) and (4.69) with

$$(4.79) \quad \left[\dot{A}_0^{(0)}, \dot{A}_1^{(0)}, \dot{B}_0^{(0)} \right] = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 4 & -2 & 2 & 18 & -12 \\ 6 & -4 & 6 & -4 & -24 & 12 \end{array} \right)$$

and the form described in (4.68) and (4.77) with

$$D_0^{-1} R_0^{-T} \left[\dot{A}_0^{(0)}, \dot{A}_1^{(0)}, \dot{B}_0^{(0)} \right] = \left(\begin{array}{cc|cc|cc} \frac{1}{6} & \frac{2}{6} & \frac{5}{6} & \frac{-2}{6} & 0 & \frac{-6}{6} \\ \frac{7}{10\sqrt{3}} & \frac{2}{10\sqrt{3}} & \frac{17}{10\sqrt{3}} & \frac{-8}{10\sqrt{3}} & \frac{-18}{10\sqrt{3}} & \frac{-6}{10\sqrt{3}} \\ \frac{5}{6\sqrt{5}} & \frac{-2}{6\sqrt{5}} & \frac{7}{6\sqrt{5}} & \frac{-4}{6\sqrt{5}} & \frac{-18}{6\sqrt{5}} & \frac{6}{6\sqrt{5}} \\ \frac{3}{10\sqrt{7}} & \frac{-2}{10\sqrt{7}} & \frac{3}{10\sqrt{7}} & \frac{-2}{10\sqrt{7}} & \frac{-12}{10\sqrt{7}} & \frac{6}{10\sqrt{7}} \end{array} \right),$$

where we recall that $\Phi_2(x) = (h_x^2, \frac{1}{2}hh_{xx})^T$. The matrices $\ddot{A}_0^{(0)}, \ddot{A}_1^{(0)}, \ddot{B}^{(0)}$ representing $\psi_{xxxx}^{(0)}$ are each 4×5 matrices while $\Phi_4(x)$ was given in (3.16).

To compute $\|\psi_{xx}^{(2k)}\|_0$ and $\|h^2\psi_{xxxx}^{(2k-2)}\|_0$, we note that each of the expressions in (4.77) is of the form $\sum_{n=0}^{2k+3} \dot{P}_n(x, y) w_n(x)$, where $w(x) = Sz(x)$, S is a constant matrix, $z(x) = h^{-\frac{m}{2}} \Phi_{2k+2j}(x)$, $j = 1$ or 2 , and $m = 1$ or 3 . For fixed x , we have $\int_0^{h(x)} \dot{P}_m(x, y) \dot{P}_n(x, y) dy = \delta_{mn}$. It follows that

$$(4.80) \quad \int_0^1 \int_0^{h(x)} \left(\sum_n \dot{P}_n(x, y) w_n(x) \right)^2 dy dx = \int_0^1 \sum_n w_n(x)^2 dx.$$

Moreover, $\sum_n w_n(x)^2 = \|w(x)\|_2^2 \leq \|S\|_F^2 \|z(x)\|_2^2$, where $\|\cdot\|_F$ and $\|\cdot\|_2$ are the Frobenius and 2-norms of a matrix and vector, respectively. Integrating, we have $\int_0^1 \|z(x)\|_2^2 dx = I_m \|\tilde{E}_m^{(2k+2j)}\|_1$. Since $\|\tilde{E}_m^{(2\ell)}\|_1 \leq d_{2\ell} r_k^{-4\ell}$ for $0 \leq \ell \leq k+1$, we define

$$(4.81) \quad \begin{aligned} K_i^{(2k)} &= \sqrt{d_{2k+2}} \left\| D_{2k}^{-1} R_{2k}^{-T} \dot{A}_i^{(2k)} \right\|_F, & i = 0, 1, \quad k \geq 0, \\ K_2^{(2k)} &= \sqrt{d_{2k+2}} \left\| D_{2k}^{-1} R_{2k}^{-T} \dot{B}^{(2k)} \right\|_F, & k \geq 0, \\ \tilde{K}_i^{(2k)} &= \sqrt{d_{2k+4}} \left\| D_{2k}^{-1} R_{2k}^{-T} \ddot{A}_i^{(2k)} \right\|_F, & i = 0, 1, \quad k \geq 0, \\ \tilde{K}_2^{(2k)} &= \sqrt{d_{2k+4}} \left\| D_{2k}^{-1} R_{2k}^{-T} \ddot{B}^{(2k)} \right\|_F, & k \geq 0, \end{aligned}$$

so that

$$(4.82) \quad \begin{aligned} \left\| \frac{I_2}{I_3} \alpha_{xx}^{(2\ell)} \right\|_0 &\leq \sqrt{I_1} \left(|V_0| K_0^{(2\ell)} + |V_1| K_1^{(2\ell)} \right) r_k^{-2\ell-2}, & 0 \leq \ell \leq k, \\ \left\| \beta_{xx}^{(2\ell)} \right\|_0 &\leq \sqrt{I_3} K_2^{(2\ell)} r_k^{-2\ell-2}, & 0 \leq \ell \leq k, \\ \left\| \frac{I_2}{I_3} h^2 \alpha_{xxxx}^{(2\ell)} \right\|_0 &\leq \sqrt{I_1} \left(|V_0| \tilde{K}_0^{(2\ell)} + |V_1| \tilde{K}_1^{(2\ell)} \right) r_k^{-2\ell-4}, & 0 \leq \ell \leq k-1, \\ \left\| h^2 \beta_{xxxx}^{(2\ell)} \right\|_0 &\leq \sqrt{I_3} \tilde{K}_2^{(2\ell)} r_k^{-2\ell-4}, & 0 \leq \ell \leq k-1. \end{aligned}$$

From the bound (4.67) on $|Q^{(2\ell)}|$ and the formula (4.68) for $\psi^{(2k)}$ in terms of $\alpha^{(2k)}$ and $\beta^{(2k-2\ell)}$, we see that after increasing $K_0^{(2k)}$, $K_1^{(2k)}$, $\tilde{K}_0^{(2k)}$, $\tilde{K}_1^{(2k)}$ via

$$(4.83) \quad \begin{aligned} K_i^{(2k)} &= K_i^{(2k)} + \sum_{\ell=0}^k \kappa_i^{(2\ell)} K_2^{(2k-2\ell)}, & i = 0, 1, \quad k \geq 0, \\ \tilde{K}_i^{(2k)} &= \tilde{K}_i^{(2k)} + \sum_{\ell=0}^k \kappa_i^{(2\ell)} \tilde{K}_2^{(2k-2\ell)}, & i = 0, 1, \quad k \geq 0, \end{aligned}$$

we have

$$(4.84) \quad \begin{aligned} \left\| \psi_{xx}^{(2k)} \right\|_0 &\leq \sqrt{I_1} \left(|V_0| K_0^{(2k)} + |V_1| K_1^{(2k)} \right) r_k^{-2k-2}, & k \geq 0, \\ \left\| h^2 \psi_{xxxx}^{(2k-2)} \right\|_0 &\leq \sqrt{I_1} \left(|V_0| \tilde{K}_0^{(2k-2)} + |V_1| \tilde{K}_1^{(2k-2)} \right) r_k^{-2k-2}, & k \geq 0. \end{aligned}$$

In each term $Q^{(2\ell)} \beta^{(2k-2\ell)}$, we have used $I_2^2/I_1 I_3 \leq 1$ (which follows from the Cauchy-Schwarz inequality) to majorize $I_2/\sqrt{I_3}$ by $\sqrt{I_1}$. The constants $K_i^{(2k)}$, $\tilde{K}_i^{(2k)}$ do not depend on h and may be computed once and for all along with the constants $\kappa_i^{(2k)}$ and the matrices $A_i^{(2k)}$ and $B^{(2k)}$; see Tables 4.2 and 4.3.

Finally, we combine the boundary estimate (4.59) with the interior estimate (4.84) to bound the truncation error via (4.48). In terms of r_k , (4.59) gives

$$(4.85) \quad \left\| \frac{\gamma_k}{h^{1/2}} \right\|_{1/2, \varepsilon} \leq |V_1| \sqrt{I_1} \left[\frac{1}{2} + \frac{\varepsilon}{r_k} \sqrt{\frac{I_3}{I_1}} \sqrt{\frac{85}{16} + \frac{20}{3}k} \right] r_k^{-2k-2}, \quad k \geq 0.$$

TABLE 4.2
 $K_i^{(2k)}$ before and after loop (4.83).

k	$K_0^{(2k)}$ before	$K_1^{(2k)}$ before	$K_2^{(2k)}$	$K_0^{(2k)}$ after	$K_1^{(2k)}$ after
0	9.95×10^{-01}	$2.17 \times 10^{+00}$	$2.99 \times 10^{+00}$	$2.49 \times 10^{+00}$	$3.67 \times 10^{+00}$
1	$2.33 \times 10^{+00}$	$4.70 \times 10^{+00}$	$7.99 \times 10^{+00}$	$8.41 \times 10^{+00}$	$1.16 \times 10^{+01}$
2	$7.42 \times 10^{+00}$	$1.58 \times 10^{+01}$	$2.51 \times 10^{+01}$	$3.14 \times 10^{+01}$	$4.32 \times 10^{+01}$
3	$4.29 \times 10^{+01}$	$8.62 \times 10^{+01}$	$1.19 \times 10^{+02}$	$1.62 \times 10^{+02}$	$2.21 \times 10^{+02}$
4	$4.58 \times 10^{+02}$	$8.71 \times 10^{+02}$	$1.03 \times 10^{+03}$	$1.34 \times 10^{+03}$	$1.87 \times 10^{+03}$
5	$7.21 \times 10^{+03}$	$1.52 \times 10^{+04}$	$1.62 \times 10^{+04}$	$1.85 \times 10^{+04}$	$2.77 \times 10^{+04}$
6	$1.87 \times 10^{+05}$	$3.54 \times 10^{+05}$	$3.51 \times 10^{+05}$	$4.06 \times 10^{+05}$	$5.90 \times 10^{+05}$
7	$6.57 \times 10^{+06}$	$1.25 \times 10^{+07}$	$1.08 \times 10^{+07}$	$1.28 \times 10^{+07}$	$1.92 \times 10^{+07}$
8	$2.74 \times 10^{+08}$	$6.17 \times 10^{+08}$	$4.64 \times 10^{+08}$	$5.31 \times 10^{+08}$	$8.87 \times 10^{+08}$
9	$1.75 \times 10^{+10}$	$3.28 \times 10^{+10}$	$2.52 \times 10^{+10}$	$3.12 \times 10^{+10}$	$4.70 \times 10^{+10}$
10	$1.32 \times 10^{+12}$	$2.40 \times 10^{+12}$	$1.69 \times 10^{+12}$	$2.22 \times 10^{+12}$	$3.34 \times 10^{+12}$

TABLE 4.3
 $\tilde{K}_i^{(2k)}$ before and after loop (4.83).

k	$\tilde{K}_0^{(2k)}$ before	$\tilde{K}_1^{(2k)}$ before	$\tilde{K}_2^{(2k)}$	$\tilde{K}_0^{(2k)}$ after	$\tilde{K}_1^{(2k)}$ after
0	$1.23 \times 10^{+02}$	$2.08 \times 10^{+02}$	$4.84 \times 10^{+02}$	$3.65 \times 10^{+02}$	$4.50 \times 10^{+02}$
1	$7.81 \times 10^{+02}$	$1.50 \times 10^{+03}$	$3.35 \times 10^{+03}$	$2.79 \times 10^{+03}$	$3.65 \times 10^{+03}$
2	$3.55 \times 10^{+03}$	$7.74 \times 10^{+03}$	$1.56 \times 10^{+04}$	$1.47 \times 10^{+04}$	$2.00 \times 10^{+04}$
3	$1.73 \times 10^{+04}$	$3.88 \times 10^{+04}$	$6.63 \times 10^{+04}$	$7.22 \times 10^{+04}$	$1.00 \times 10^{+05}$
4	$1.85 \times 10^{+05}$	$3.76 \times 10^{+05}$	$4.38 \times 10^{+05}$	$5.37 \times 10^{+05}$	$7.68 \times 10^{+05}$
5	$3.97 \times 10^{+06}$	$8.59 \times 10^{+06}$	$8.12 \times 10^{+06}$	$9.04 \times 10^{+06}$	$1.40 \times 10^{+07}$
6	$1.45 \times 10^{+08}$	$2.66 \times 10^{+08}$	$2.46 \times 10^{+08}$	$2.81 \times 10^{+08}$	$4.08 \times 10^{+08}$
7	$6.62 \times 10^{+09}$	$1.26 \times 10^{+10}$	$1.00 \times 10^{+10}$	$1.19 \times 10^{+10}$	$1.81 \times 10^{+10}$
8	$3.50 \times 10^{+11}$	$8.00 \times 10^{+11}$	$5.50 \times 10^{+11}$	$6.36 \times 10^{+11}$	$1.09 \times 10^{+12}$
9	$2.81 \times 10^{+13}$	$5.24 \times 10^{+13}$	$3.81 \times 10^{+13}$	$4.76 \times 10^{+13}$	$7.22 \times 10^{+13}$
10	$2.59 \times 10^{+15}$	$4.70 \times 10^{+15}$	$3.12 \times 10^{+15}$	$4.18 \times 10^{+15}$	$6.31 \times 10^{+15}$

We now define

$$(4.86) \quad \rho_k = \begin{cases} \left[\max \left(5K_0^{(2k)}, 5K_1^{(2k)} + \frac{15}{2} \right) \right]^{-\frac{1}{2k+2}}, & k = 0, \\ \left[\max \left(5K_0^{(2k)} + \tilde{K}_0^{(2k-2)}, 5K_1^{(2k)} + \tilde{K}_1^{(2k-2)} + \frac{15}{2}, \rho_{k-1}^{-2k} \right) \right]^{-\frac{1}{2k+2}}, & k \geq 1, \end{cases}$$

so that (4.49), (4.84), and (4.85) imply

$$(4.87) \quad \begin{aligned} & \left\| \psi_{err}^{(2k)} \right\|_{2,\varepsilon} + \varepsilon^{2k+2} \left\| \psi_{xx}^{(2k)} \right\|_0 \\ & \leq \sqrt{I_1} (|V_0| + |V_1|) \left[\rho_k^{-2k-2} + 15 \frac{\varepsilon}{r_k} \sqrt{\frac{I_3}{I_1}} \sqrt{\frac{85}{16} + \frac{20}{3}k} \right] \left(\frac{\varepsilon}{r_k} \right)^{2k+2}. \end{aligned}$$

To simplify this expression, we define

$$(4.88) \quad \theta_k = 15 \rho_k^{2k+2} \sqrt{\frac{85}{16} + \frac{20}{3}k}$$

and summarize the main result of this section as a theorem.

TABLE 4.4
 ρ_k and θ_k .

k	ρ_k	θ_k
0	0.197	$1.34 \times 10^{+00}$
1	0.210	1.01×10^{-01}
2	0.252	1.67×10^{-02}
3	0.288	3.58×10^{-03}
4	0.313	7.73×10^{-04}
5	0.319	1.03×10^{-04}
6	0.305	5.96×10^{-06}
7	0.286	2.15×10^{-07}
8	0.266	5.10×10^{-09}
9	0.248	9.15×10^{-11}
10	0.232	1.43×10^{-12}
11	0.218	1.69×10^{-14}
12	0.204	1.58×10^{-16}
13	0.193	1.42×10^{-18}
14	0.183	1.04×10^{-20}
15	0.173	5.98×10^{-23}
16	0.164	3.46×10^{-25}
17	0.157	1.75×10^{-27}
18	0.149	6.86×10^{-30}
19	0.143	2.72×10^{-32}
20	0.137	1.02×10^{-34}
21	0.131	2.94×10^{-37}
22	0.126	8.36×10^{-40}
23	0.122	2.41×10^{-42}
24	0.117	5.40×10^{-45}
25	0.113	1.15×10^{-47}

THEOREM 4.8. Suppose $k \geq 0$, $h \in C^{2k+1,1}(T)$, $0 < h(x) \leq 1$ for $0 \leq x \leq 1$, and $\varepsilon \leq r_0/3$. Then the truncation errors $\psi_{err}^{(2k)}$ and $Q_{err}^{(2k)}$ in (4.37) satisfy the bound

$$\begin{aligned}
 \sqrt{3} \left| Q_{err}^{(2k)} \right| &\leq \left\| \psi_{err}^{(2k)} \right\|_{2,\varepsilon} \leq \left\| \psi_{err}^{(2k)} \right\|_{2,\varepsilon} + \varepsilon^{2k+2} \left\| \psi_{xx}^{(2k)} \right\|_0 \\
 (4.89) \qquad &\leq \sqrt{I_1} (|V_0| + |V_1|) \left[1 + \theta_k \frac{\varepsilon}{r_k} \sqrt{\frac{I_3}{I_1}} \right] \left(\frac{\varepsilon}{\rho_k r_k} \right)^{2k+2},
 \end{aligned}$$

where I_m , r_k , ρ_k , and θ_k were defined in (2.9), (4.61), (4.86), and (4.88).

Remark 4.9. The constants in this estimate have been organized to be either (1) given in the problem statement or easily computable from h ; or (2) difficult to compute but universal (independent of h). The first 26 constants in the latter category (ρ_k and θ_k) are given in Table 4.4. We have therefore identified the features of h that are most likely to affect the validity of the lubrication approximation. In particular, higher derivatives are allowed to be large in regions where h is small (since r_k depends on the uniform norms of the products $\frac{1}{\ell!} h^{\ell-1} \partial_x^\ell h$ rather than on $\partial_x^\ell h$ alone).

Remark 4.10. The term ρ_{k-1}^{-2k} in the definition of ρ_k ensures that ρ_k^{2k+2} is a nonincreasing function of k . This assumption is useful in the following section when deriving a bound on the truncation error of the pressure. Note that ρ_k itself is allowed to increase as long as ρ_k^{2k+2} does not. It is probably not necessary to include this term in the definition of ρ_k since it is not the argmax for $1 \leq k \leq 25$, and by that point θ_k (and hence ρ_k^{2k+2}) appears to be decreasing rapidly without it; see Figure 4.1.

4.5. Velocity, vorticity, and pressure. We now show how to use the error bound we have obtained for the stream function to bound the error in the velocity,

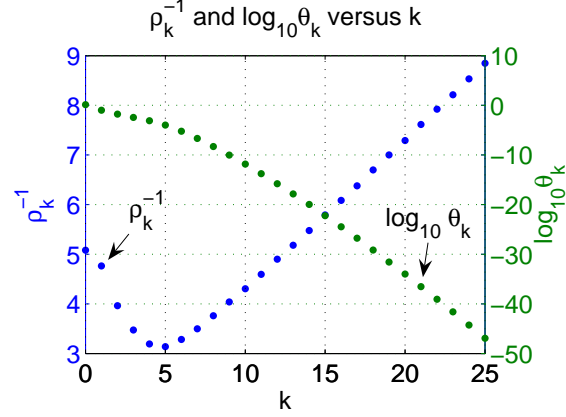


FIG. 4.1. Plot of ρ_k^{-1} and $\log_{10} \theta_k$ versus k . Note that ρ_k^{-1} initially decreases but eventually grows almost linearly, indicating that the lubrication expansion is probably an asymptotic series rather than a convergent series. The term involving θ_k in (4.89) is only important when k is small, since θ_k converges rapidly to zero as $k \rightarrow \infty$.

vorticity, and pressure. We define $u^{(2k)}$, $v^{(2k)}$, $\omega^{(2k)}$, and $p^{(2k)}$ in terms of $\psi^{(2k)}$ as in (3.8) and define, e.g.,

$$(4.90) \quad \omega_{err}^{(2k)} = \omega_{exact} - \omega_{approx}^{(2k)}, \quad \omega_{approx}^{(2k)} = \omega^{(0)} + \varepsilon^2 \omega^{(2)} + \dots + \varepsilon^{2k} \omega^{(2k)}.$$

From (3.8), we then have

$$(4.91) \quad (u_{err}^{(2k)}, v_{err}^{(2k)}) = \nabla \times \psi_{err}^{(2k)}, \quad \omega_{err}^{(2k)} = -\Delta_\varepsilon \psi_{err}^{(2k)} - \varepsilon^{2k+2} \psi_{xx}^{(2k)},$$

$$(4.92) \quad \partial_x p_{err}^{(2k)} = -\partial_y \omega_{err}^{(2k)}, \quad \partial_y p_{err}^{(2k)} = \begin{cases} \partial_x (\varepsilon^2 \omega_{exact}), & k = 0, \\ \partial_x (\varepsilon^2 \omega_{err}^{(2k-2)}), & k \geq 1, \end{cases}$$

which immediately gives bounds on the error in velocity and vorticity:

$$(4.93) \quad \left(\|u_{err}^{(2k)}\|_{1,\varepsilon}^2 + \|\varepsilon v_{err}^{(2k)}\|_{1,\varepsilon}^2 \right)^{1/2} \leq \|\psi_{err}^{(2k)}\|_{2,\varepsilon} \leq (*), \quad k \geq 0, \\ \|\omega_{err}^{(2k)}\|_0 \leq \|\psi_{err}^{(2k)}\|_{2,\varepsilon} + \varepsilon^{2k+2} \|\psi_{xx}^{(2k)}\|_0 \leq (*), \quad k \geq 0,$$

where $(*)$ is the right-hand side of (4.89). Obtaining a bound on the error in the pressure is somewhat more difficult, as it relies on the fact that the gradient is an isomorphism from $L_{\#}^2(\Omega)$ (the space of square integrable functions with zero mean) onto the polar set

$$(4.94) \quad V^0 = \{\mathbf{f} \in H^{-1}(\Omega)^2 : \langle \mathbf{f}, \mathbf{u} \rangle = 0 \text{ whenever } \mathbf{u} \in V\},$$

where $V = \{\mathbf{u} \in H_0^1(\Omega)^2 : \nabla \cdot \mathbf{u} = 0\}$. Given $\mathbf{f} \in V^0$, there is a unique $p \in L_{\#}^2(\Omega)$ such that $\nabla p = \mathbf{f}$; moreover, p satisfies

$$(4.95) \quad \|p\|_0 \leq \beta^{-1} \|\mathbf{f}\|_{-1}, \quad \beta = \inf_p \sup_{\mathbf{u}} \frac{|(p, \nabla \cdot \mathbf{u})|}{\|p\|_0 \|\mathbf{u}\|_1}.$$

Here we use a standard (unweighted) Sobolev norm for \mathbf{f} . More precisely, as $\|\cdot\|_1$ and $|\cdot|_1$ are equivalent on $H_0^1(\Omega)$, the negative norms

$$(4.96) \quad |\mathbf{f}|_{-1} = \sup_{|u|_1^2 + |v|_1^2 = 1} |\langle \mathbf{f}, (u; v) \rangle|, \quad \|\mathbf{f}\|_{-1} = \sup_{\|u\|_1^2 + \|v\|_1^2 = 1} |\langle \mathbf{f}, (u; v) \rangle|$$

are equivalent on $H^{-1}(\Omega)^2$. Since $|u|_1 \leq \|u\|_1$ for all $u \in H_0^1(\Omega)$, we have $\|\mathbf{f}\|_{-1} \leq |\mathbf{f}|_{-1}$ for all $\mathbf{f} \in H^{-1}(\Omega)^2$.

Explicit estimates [7, 23, 10] for the LBB constant β in (4.95) have been obtained for rectangular domains (with no periodicity), e.g.,

$$(4.97) \quad \frac{1}{\ell} \sin \frac{\pi}{8} \leq \beta \leq \frac{\pi}{2\sqrt{3}\ell}, \quad \ell = \max \left(\frac{L_1}{L_2}, \frac{L_2}{L_1} \right), \quad \Omega = (0, L_1) \times (0, L_2).$$

The lower bound here also works for an x -periodic rectangle as the condition that $\mathbf{u}|_{x=0} = 0$ and $\mathbf{u}|_{x=L_1} = 0$ is more restrictive than $\mathbf{u}|_{x=0} = \mathbf{u}|_{x=L_1}$. Explicit estimates are also known for domains that are star-shaped with respect to each point in a ball of radius R contained inside Ω ; see [13, 23]. Our interest in the present work is in x -periodic domains with the upper boundary given by a function $h(x)$. Such domains are not in general star-shaped, so the previously known results do not apply. In [24], we improve the estimate for the lower bound on β in (4.97) for an x -periodic rectangle by a factor of about 3.5 and show how to avoid invoking Rellich's theorem in the change of variables to the case that Ω is x -periodic with the upper boundary given by $h(x)$. It is shown that

$$\beta^{-1} \leq \frac{9}{4} (1 + r_0^{-2}) \left(\frac{h_1}{h_0} \right)^{1/2} \max \left(4, \frac{L}{h_0}, \frac{h_1}{h_0} \right), \quad \begin{aligned} h_0 &= \min_{0 \leq x \leq L} h(x), \\ h_1 &= \max_{0 \leq x \leq L} h(x), \end{aligned}$$

where $r_0 = \max(\|h_x\|_\infty, \|\frac{1}{2} h h_{xx}\|_\infty^{1/2})^{-1}$ and L is the period of $h(x)$. In the current case, the length scales \bar{H} and \bar{W} were chosen so that $L = 1$ and $h_1 \leq 1$ in the dimensionless problem. Thus, solving $\nabla p = \mathbf{f}$ yields

$$(4.98) \quad \|p\|_0 \leq \beta^{-1} |\mathbf{f}|_{-1}, \quad \beta^{-1} \leq \max \left(9h_0^{-1/2}, \frac{9}{4} h_0^{-3/2} \right) (1 + r_0^{-2}).$$

The dependence on gap thickness h_0 occurs because p can change rapidly in the gap without a large penalty from \mathbf{u} . In [24], an example is given to show that the factor of $h_0^{-3/2}$ in the formula (4.98) for β^{-1} cannot be improved.

We have reduced the problem of bounding $p_{err}^{(2k)}$ to that of bounding the functional on the right-hand side of $\nabla p_{err}^{(2k)} = \mathbf{f}_k$ in (4.92), namely,

$$(4.99) \quad \langle \mathbf{f}_k, (u; v) \rangle = \begin{cases} \int_\Omega \omega_{err}^{(0)} u_y - \varepsilon^2 \omega_{exact} v_x dA, & k = 0, \\ \int_\Omega \omega_{err}^{(2k)} u_y - \varepsilon^2 \omega_{err}^{(2k-2)} v_x dA, & k \geq 1. \end{cases}$$

First, we check that \mathbf{f}_k belongs to V^0 . If $u, v \in H_0^1(\Omega)$ and $u_x + v_y = 0$, the function $\phi(x, y) = \int_0^y u(x, \eta) d\eta$ satisfies $\phi_y = u$, $\phi_x = -v$ and so belongs to Ψ . As a result,

$$(4.100) \quad \begin{aligned} \langle \mathbf{f}_0, (u; v) \rangle &= \int_\Omega -\omega^{(0)} \phi_{yy} + \omega_{exact} (\phi_{yy} + \varepsilon^2 \phi_{xx}) dA \\ &= \int_\Omega \psi_{yy}^{(0)} \phi_{yy} + (\Delta_\varepsilon \psi_{exact})(\Delta_\varepsilon \phi) dA = 0, \end{aligned}$$

and for $k \geq 1$,

$$\begin{aligned} \langle \mathbf{f}_k, (u; v) \rangle &= \overbrace{\langle \mathbf{f}_{k-1}, (u; v) \rangle}^0 + \int_{\Omega} -\varepsilon^{2k} \omega^{(2k)} \phi_{yy} - \varepsilon^{2k} \omega^{(2k-2)} \phi_{xx} dA \\ &= \varepsilon^{2k} \int_{\Omega} \underbrace{\psi_{yy}^{(2k)} \phi_{yy} + \psi_{xx}^{(2k-2)} \phi_{yy} + \psi_{yy}^{(2k-2)} \phi_{xx} + \psi_{xx}^{(2k-4)} \phi_{xx}}_{\text{omit if } k=1} dA = 0. \end{aligned}$$

Next, we bound the norm of \mathbf{f}_k . From (4.99), we see that

$$|\mathbf{f}_0|_{-1} \leq \left\| \omega_{err}^{(0)} \right\|_0 + \left\| \varepsilon^2 \omega_{exact} \right\|_0, \quad |\mathbf{f}_k|_{-1} \leq \left\| \omega_{err}^{(2k)} \right\|_0 + \left\| \varepsilon^2 \omega_{err}^{(2k-2)} \right\|_0, \quad k \geq 1.$$

Denoting the right-hand side of (4.89) by $(*)$, we claim that

$$(4.101) \quad \frac{\varepsilon^2}{r_0^2} \left\| \omega_{exact} \right\|_0 \leq (*), \quad (k=0), \quad \frac{\varepsilon^2}{r_k^2} \left\| \omega_{err}^{(2k-2)} \right\|_0 \leq (*), \quad (k \geq 1).$$

Once this is shown to be true, we will have the following bound on $|\mathbf{f}_k|_{-1}$:

$$(4.102) \quad |\mathbf{f}_k|_{-1} \leq (1 + r_k^2) (*), \quad (k \geq 0).$$

Together with (4.98) and the fact that $r_k \leq r_0$ for $k \geq 0$, this will give

$$(4.103) \quad \left\| p_{err}^{(2k)} \right\|_0 \leq \max \left(9h_0^{-1/2}, \frac{9}{4}h_0^{-3/2} \right) (r_k + r_k^{-1})^2 (*).$$

Note that there is at least a power of r_k^{-2} in $(*)$ to prevent this bound from diverging as $r_k \rightarrow \infty$. It may be possible to improve the bound in this regime by replacing $\left\| \varepsilon^2 \omega_{exact} \right\|_0$ in (4.101) by $\left\| \varepsilon^2 \partial_x \omega_{exact} \right\|_{-1}$, but this seems very difficult. At any rate, if $r_k = \infty$, then $(*) = 0$, $h(x)$ is a constant function, the exact and approximate vorticity are constants, \mathbf{f}_k is the zero functional, and $p_{exact} = p_{err}^{(2k)} = 0$. Let us now prove (4.101). For $k \geq 1$, this follows from

$$(4.104) \quad \begin{aligned} &\left[\rho_{k-1}^{-2k} + 15 \frac{\varepsilon}{r_{k-1}} \sqrt{\frac{I_3}{I_1}} \sqrt{\frac{85}{16} + \frac{20}{3}(k-1)} \right] \left(\frac{\varepsilon}{r_{k-1}} \right)^{2k} \left(\frac{\varepsilon}{r_k} \right)^2 \\ &\leq \left[\rho_k^{-2k-2} + 15 \frac{\varepsilon}{r_k} \sqrt{\frac{I_3}{I_1}} \sqrt{\frac{85}{16} + \frac{20}{3}k} \right] \left(\frac{\varepsilon}{r_k} \right)^{2k+2}, \end{aligned}$$

which holds because ρ_k^{2k+2} and r_k are nonincreasing functions of k ; see Remark 4.10 and the definition of r_k in (4.61). For $k=0$, we use Theorem 4.4 to conclude that

$$(4.105) \quad \left\| \omega_{exact} \right\|_0 \leq \left\| \psi \right\|_{2,\varepsilon} \leq 15 \left(\left\| h^{-1/2} g_0 \right\|_{1/2,\varepsilon} + \left\| h^{-1/2} g_1 \right\|_{1/2,\varepsilon} \right),$$

where $g_0(x) = V_0$ and $g_1(x) = (1 + \varepsilon^2 h_x(x)^2)^{-1/2}$. Now

$$(4.106) \quad \begin{aligned} \left\| h^{-1/2} g_0 \right\|_{1/2,\varepsilon}^2 &\leq V_0^2 \left\| h^{-1/2} \right\|_{1,\varepsilon}^2 = V_0^2 \int_0^1 h^{-1} + \frac{1}{4} (\varepsilon h_x)^2 h^{-3} dx \\ &\leq V_0^2 \left[I_1 + \frac{1}{4} \left(\frac{\varepsilon}{r_0} \right)^2 I_3 \right] \end{aligned}$$

and

(4.107)

$$\begin{aligned} \left\| h^{-1/2} g_1 \right\|_{1/2, \varepsilon}^2 &\leq V_1^2 \left\| h^{-1/2} (1 + \varepsilon^2 h_x)^{-1/2} \right\|_{1, \varepsilon}^2 \\ &= V_1^2 \int_0^1 (h^{-1}) (\cdot)^{-1} + \left[-\frac{1}{2} h^{-3/2} (\varepsilon h_x) (\cdot)^{-1/2} - (h^{-1/2}) (\cdot)^{-3/2} \varepsilon^3 h_x h_{xx} \right]^2 dx \\ &\leq V_1^2 \int_0^1 h^{-1} + \frac{1}{2} h^{-3} (\varepsilon h_x)^2 + 8 h^{-3} (\varepsilon h_x)^2 \left(\varepsilon^2 \frac{h h_{xx}}{2} \right)^2 dx \end{aligned}$$

(4.108)

$$\leq V_1^2 \left[I_1 + \left(\frac{1}{2} + 8 \frac{\varepsilon^4}{r_0^4} \right) \left(\frac{\varepsilon}{r_0} \right)^2 I_3 \right].$$

Since we have assumed that $\varepsilon \leq r_0/3$, we conclude that

$$(4.109) \quad \frac{\varepsilon^2}{r_0^2} \|\omega_{exact}\|_0 \leq \sqrt{I_1} (|V_0| + |V_1|) \left[15 + \left(15 \sqrt{\frac{1}{2} + \frac{8}{81}} \right) \frac{\varepsilon}{r_0} \sqrt{\frac{I_3}{I_1}} \right] \frac{\varepsilon^2}{r_0^2}.$$

Comparing this to (4.87) with $k = 0$ and noting from Table 4.4 that $\rho_0^{-2} \geq 15$, we obtain (4.101) as claimed. Thus, we have proved the following theorem.

THEOREM 4.11. *Suppose $k \geq 0$, $h \in C^{2k+1,1}(T)$, $0 < h_0 \leq h(x) \leq 1$ for $x \in T$, and $\varepsilon \leq r_0/3$. Then the truncation errors of the stream function, flux, velocity, vorticity, and pressure satisfy the bounds*

$$(4.110) \quad \begin{aligned} \left\| \psi_{err}^{(2k)} \right\|_{2, \varepsilon} &\leq (*), \quad |Q_{err}^{(2k)}| \leq \frac{(*)}{\sqrt{3}}, \quad \left(\left\| u_{err}^{(2k)} \right\|_{1, \varepsilon}^2 + \left\| \varepsilon v_{err}^{(2k)} \right\|_{1, \varepsilon}^2 \right)^{1/2} \leq (*), \\ \left\| \omega_{err}^{(2k)} \right\|_0 &\leq (*), \quad \left\| p_{err}^{(2k)} \right\|_0 \leq \max \left(9 h_0^{-1/2}, \frac{9}{4} h_0^{-3/2} \right) (r_k + r_k^{-1})^2 (*), \end{aligned}$$

where $T = [0, 1]_p$ is the periodic unit interval

$$(4.111) \quad (*) = \sqrt{I_1} (|V_0| + |V_1|) \left[1 + \theta_k \frac{\varepsilon}{r_k} \sqrt{\frac{I_3}{I_1}} \right] \left(\frac{\varepsilon}{\rho_k r_k} \right)^{2k+2},$$

$$(4.112) \quad r_k = \left(\max_{1 \leq \ell \leq 2k+2} \left\{ \left\| \frac{1}{\ell!} h^{\ell-1} \partial_x^\ell h \right\|_\infty^{1/\ell} \right\} \right)^{-1}, \quad I_m = \int_0^1 h(x)^{-m} dx,$$

and ρ_k, θ_k are constants independent of h that can be computed once and for all as described in section 4.4 and listed in Table 4.4.

5. Finite element validation. In this section, to test the error bounds of Theorem 4.11, we compute $u_{err}^{(2k)}, v_{err}^{(2k)}, p_{err}^{(2k)}, \omega_{err}^{(2k)}$ numerically for the simple geometry described by

$$(5.1) \quad h(x) = \frac{1+a}{2} + \frac{1-a}{2} \sin(2\pi x), \quad \begin{aligned} \text{Case 1: } &a = 1/5, \\ \text{Case 2: } &a = 1/100, \end{aligned}$$

with boundary conditions $V_0 = -0.5, V_1 = 1$. We do this by comparing $u_{approx}^{(2k)}, v_{approx}^{(2k)}, p_{approx}^{(2k)}, \omega_{approx}^{(2k)}$ in (4.90) to finite element solutions of the Stokes equations on appropriately rescaled geometries.

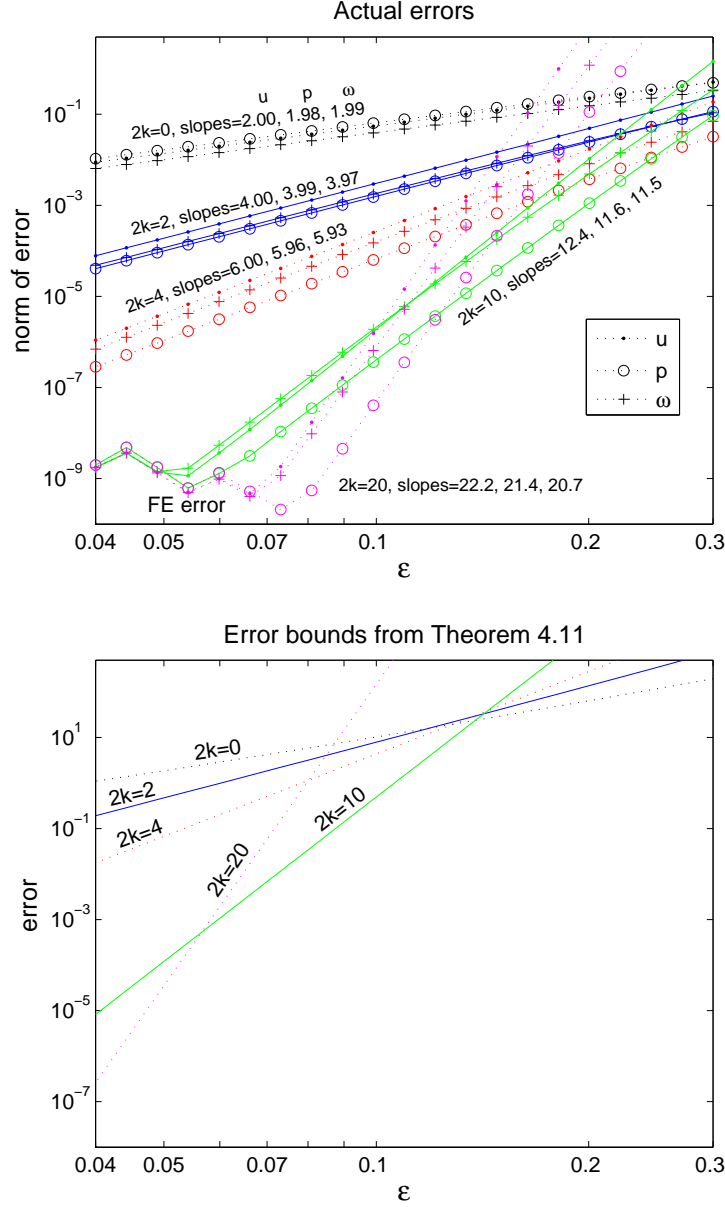


FIG. 5.1. *Top: plot of $\|\mathbf{u}_{err}^{(2k)}\|_{1,\epsilon} = (\|u_{err}^{(2k)}\|_{1,\epsilon}^2 + \|\varepsilon v_{err}^{(2k)}\|_{1,\epsilon}^2)^{1/2}$, $\|p_{err}^{(2k)}\|_0$, and $\|\omega_{err}^{(2k)}\|_0$ for $a = 1/5$, $0.04 \leq \epsilon \leq 0.3$, and $2k \in \{0, 2, 4, 10, 20\}$. The slopes of the lines were computed via linear regression using the smallest 10 values of ϵ for which the finite element solution is trusted ($\epsilon \geq 0.066$ for $2k = 10$ and $\epsilon \geq 0.09$ for $2k = 20$). As expected, for fixed k , the error is $O(\epsilon^{2k+2})$. Bottom: plot of the error bound (*) in (4.111), using $V_0 = -0.5$, $V_1 = 1$, $I_1 = 2.236$, $I_3 = 24.60$, and $r_k = 0.3559$ for $k \geq 0$, as appropriate for $h(x)$ in (5.1) with $a = 1/5$.*

The results are summarized in Figures 5.1–5.5. For 21 values of ϵ spaced exponentially between $\epsilon_0 = 0.04$ and $\epsilon_{20} = 0.3$, we set up a logically rectangular, $M \times N$ finite element mesh on the domain

$$(5.2) \quad \Omega_\epsilon = \{(x, y) : 0 \leq x \leq 1, 0 < y < \epsilon h(x)\}.$$

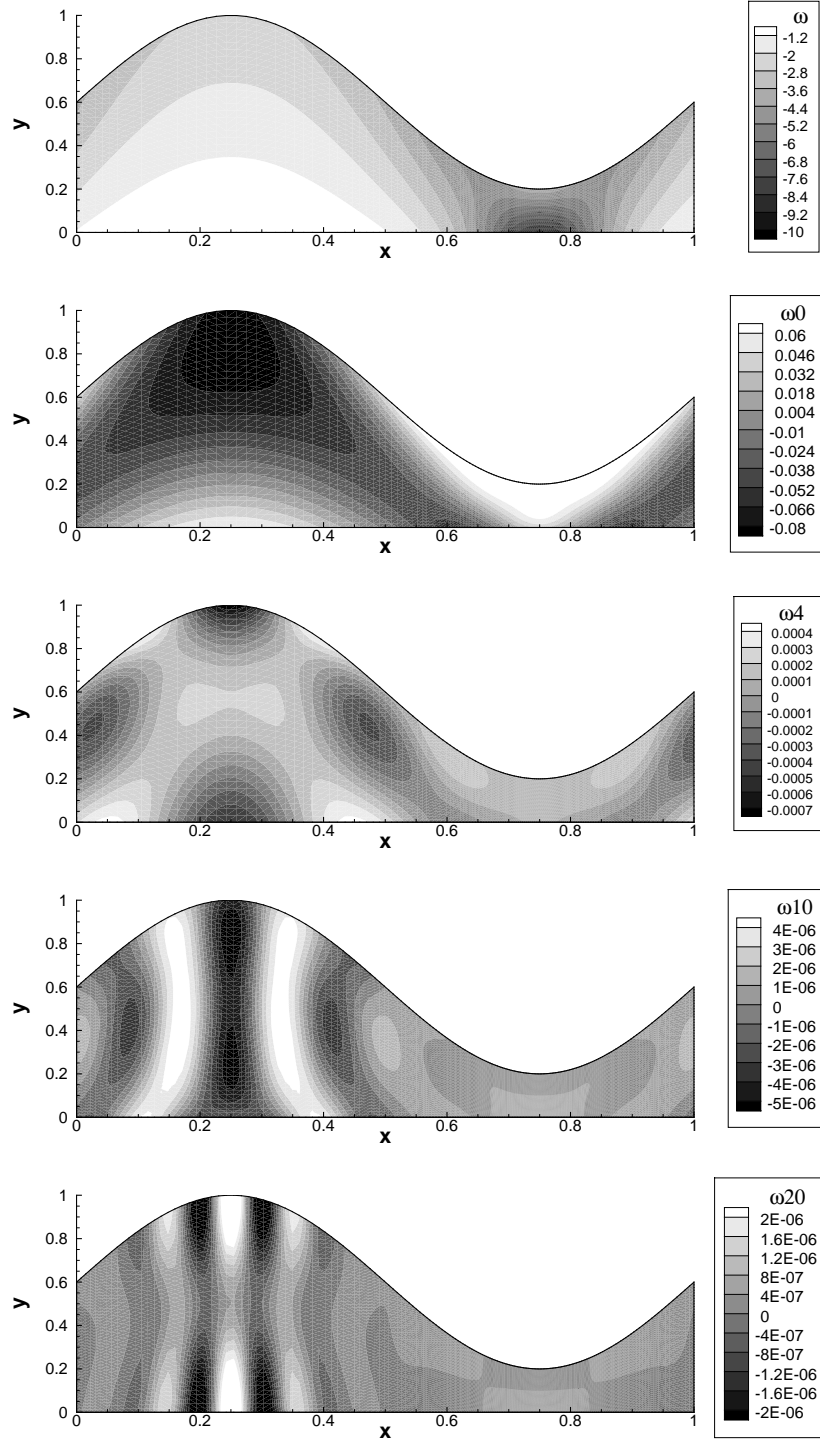


FIG. 5.2. Contour plots of ω_{exact} , $\omega_{err}^{(0)}$, $\omega_{err}^{(4)}$, $\omega_{err}^{(10)}$, and $\omega_{err}^{(20)}$ for $h(x)$ in (5.1) with $a = 1/5$, $V_0 = -0.5$, $V_1 = 1.0$, and $\varepsilon = 0.099$. Each of these plots corresponds to one of the markers in Figure 5.1.

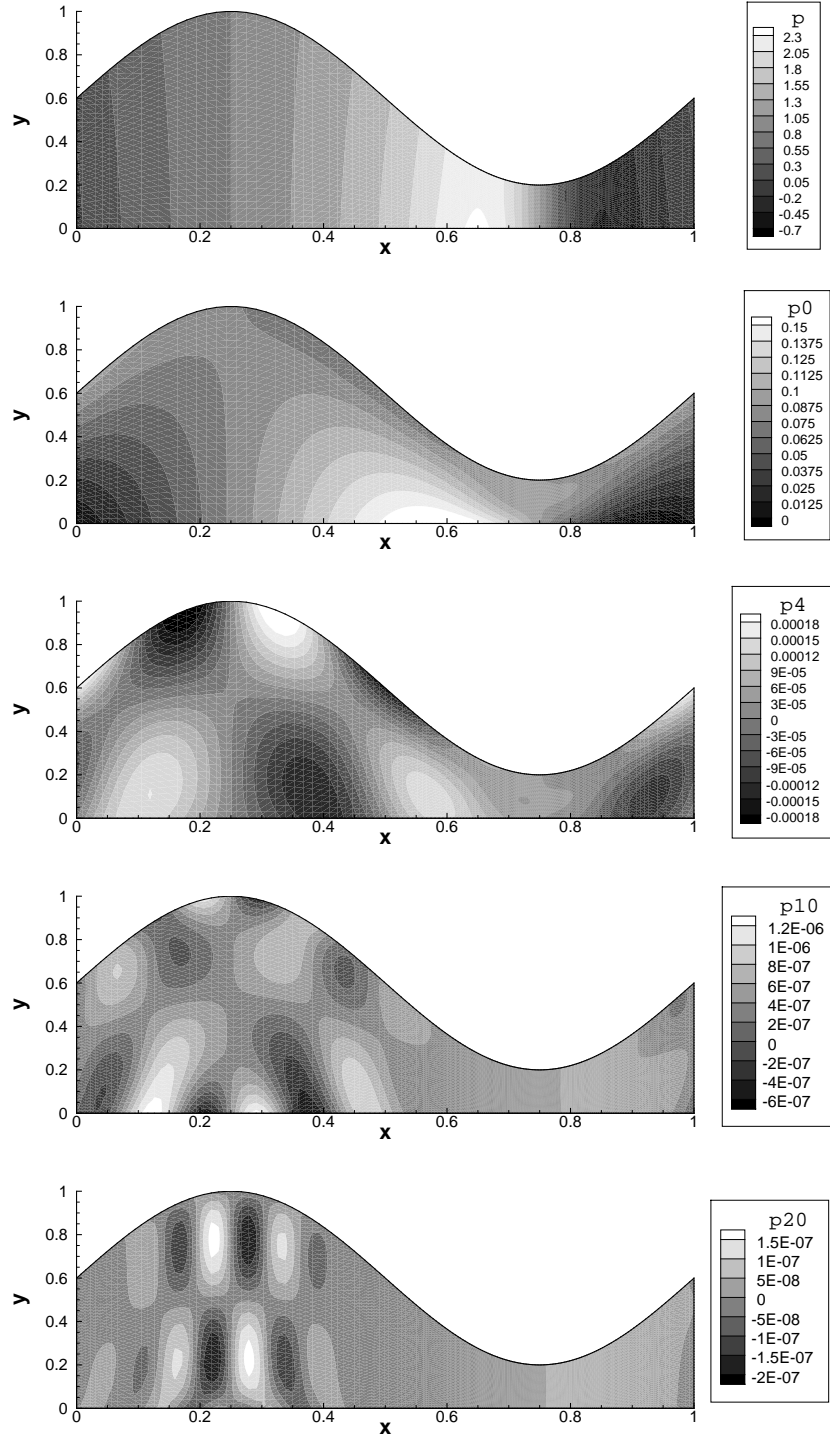
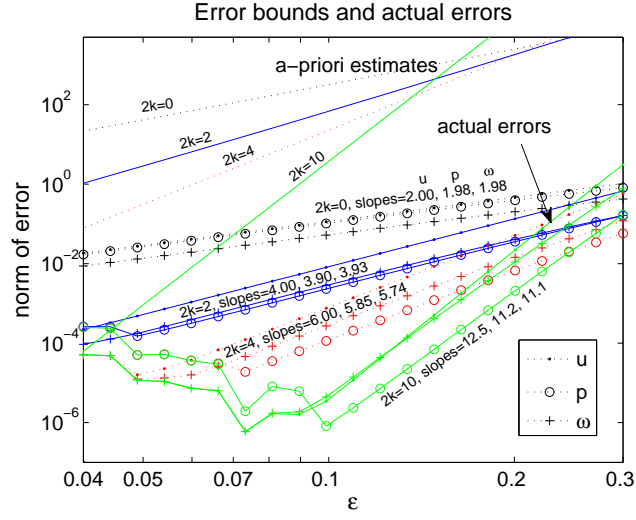
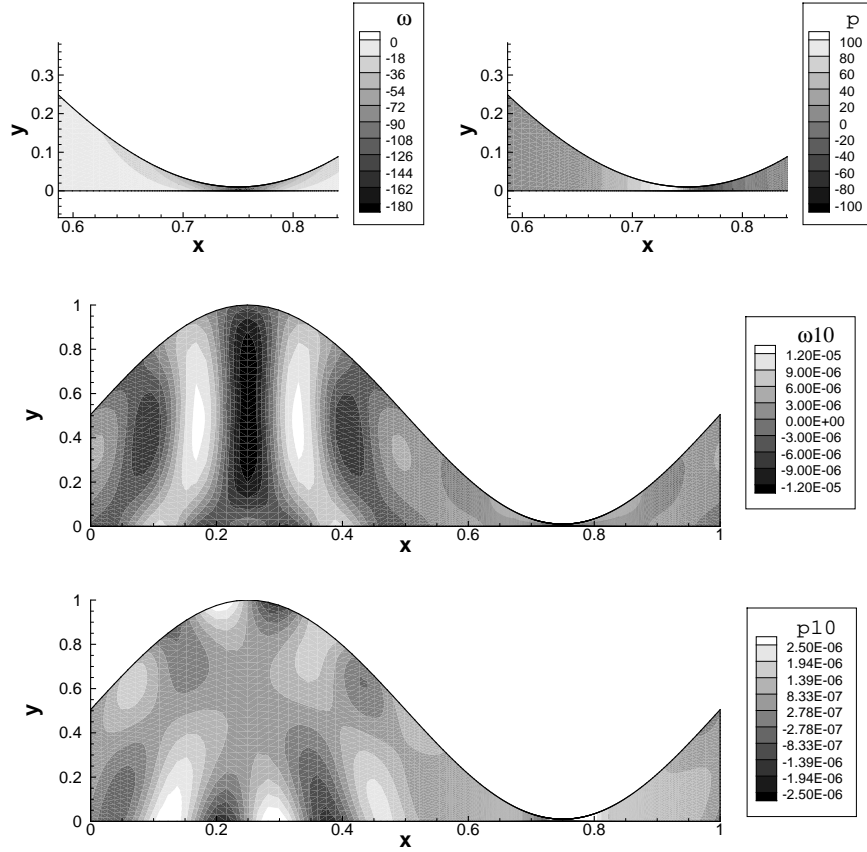


FIG. 5.3. Contour plots of p_{exact} , $p_{\text{err}}^{(0)}$, $p_{\text{err}}^{(4)}$, $p_{\text{err}}^{(10)}$, and $p_{\text{err}}^{(20)}$ for $h(x)$ in (5.1) with $a = 1/5$ and $\varepsilon = 0.099$. The “exact” solution was computed using a least squares finite element method with 15 node quartic triangular elements on a 2208×96 grid.

FIG. 5.4. Error estimates and actual errors with $a = 1/100$.FIG. 5.5. Plots of ω_{exact} , p_{exact} , $\omega_{\text{err}}^{(10)}$, and $p_{\text{err}}^{(10)}$ with $a = 1/100$, $N = 64$, $M = 4800$.

The mesh points are aligned vertically with equal spacing $\Delta y = h(x)/N$, while the grid spacing in the x -direction is chosen to keep the aspect ratios of the grid cells as close to 1 as possible; we do this by solving an ODE to enforce $\Delta x \approx h(x)/N$, which also determines M . For $a = 1/5$, we use $N = 96$ with M ranging from 768 to 5376 as ε ranges from 0.3 to 0.04; for $a = 1/100$, we use $N = 64$ with M ranging from 1600 to 10368. Four-by-four blocks of neighboring grid cells are merged and cut into two 15 node triangles. Interior nodes of the triangles are adjusted to keep the edges straight except on the top boundary, where we use quartic isoparametric elements. We solve the Stokes equations on this mesh using a least squares finite element method similar to [6] but using quartic elements to model the velocity components u and v , the pressure p , the vorticity $\omega = v_x - u_y$, and two strain rates $\tau = u_y + v_x$ and $\gamma = v_y - u_x$. We use multigrid to solve the resulting system of equations, which takes from 3 to 15 minutes on a 2.4 GHz desktop machine with 16 GB RAM.

Once the finite element solution is known at the grid points, we normalize the velocity, pressure, and vorticity as described in section 2 and rescale the domain from Ω_ε to Ω . We then use the method described in Appendix A to compute $\psi^{(0)}$, $\psi^{(2)}, \dots, \psi^{(20)}$ and their derivatives through order 3 at the grid points. Next, we use the formulas in (3.8) to obtain $u^{(2k)}$, $v^{(2k)}$, $\omega^{(2k)}$, and $p^{(2k)}$ for $k = 0, \dots, 10$. For pressure, we use 20 point Gaussian quadrature to integrate $p_x^{(2k)}$ along the x -axis to determine $p^{(2k)}(x, 0)$ at the mesh nodes. The integration of $p_y^{(2k)}$ in the y -direction is done analytically. With the expansion coefficients in hand, we evaluate

$$(5.3) \quad u_{err}^{(2k)} = u_{exact} - u_{approx}^{(2k)}, \quad u_{approx}^{(2k)} = u^{(0)} + \varepsilon^2 u^{(2)} + \dots + \varepsilon^{2k} u^{(2k)},$$

etc., at the grid nodes, where we use the finite element solution for u_{exact} . We then run through the triangles and sum up the local contributions to the errors

$$(5.4) \quad \left\| u_{err}^{(2k)} \right\|_{1,\varepsilon}^2 + \left\| \varepsilon v_{err}^{(2k)} \right\|_{1,\varepsilon}^2, \quad \left\| p_{err}^{(2k)} \right\|_0^2, \quad \left\| \omega_{err}^{(2k)} \right\|_0^2$$

by interpolating the values at the grid nodes and integrating the resulting polynomials on the triangle; this step is very similar to the assembly of the stiffness matrix. Finally, we store the results in a file for visualization (see Figures 5.2, 5.3, and 5.5) and record the norms of the truncation errors for comparison with the error bounds of Theorem 4.11.

The results of this comparison are shown in Figures 5.1 and 5.4. As expected, for fixed k , the actual errors decay as $O(\varepsilon^{2k+2})$. The a priori error bounds eventually decrease like $O(\varepsilon^{2k+2})$ as well, but the term involving θ_k in (4.111) is significant over this range of ε in some of the cases, causing the slopes to be larger:

$$\frac{\theta_k}{r_k} \sqrt{\frac{I_3}{I_1}} = \left\{ \begin{array}{c|c|c|c|c|c} & k=0 & k=1 & k=2 & k=5 & k=10 \\ \hline a=1/5 & 12.5 & 0.94 & 0.16 & .00096 & 1.3 \times 10^{-11} \\ \hline a=1/100 & 257 & 19.3 & 3.2 & 0.020 & 2.7 \times 10^{-10} \end{array} \right.$$

This effect is much more pronounced when $a = 1/100$ in (5.1) due to

$$(5.5) \quad \sqrt{\frac{I_3}{I_1}} = \frac{1}{2} \sqrt{\frac{3}{2} + \frac{1}{a} + \frac{3}{2a^2}} = \begin{cases} 3.32, & a = 1/5, \\ 61.4, & a = 1/100. \end{cases}$$

The deviation from linearity in the plots of “actual error” for small ε and large k is due to error in the finite element solutions, which are accurate to about 9 digits. This

occurs sooner when $a = 1/100$ since the pressure and vorticity of the exact solution in the vicinity of the narrow gap increases as a decreases, and also because we were forced to use a coarser mesh with $a = 1/100$ to avoid running out of computer memory in the finite element simulations. The data points with $\varepsilon = 0.099$ in Figure 5.1 correspond to the contour plots in Figures 5.2 and 5.3, where we plot ω_{exact} , $\omega_{err}^{(2k)}$, p_{exact} , and $p_{err}^{(2k)}$ for $2k = 0, 4, 10, 20$. The data points with $\varepsilon = 0.099$ in Figure 5.4 correspond to the contour plots in Figure 5.5. We remark that the apparently large value of $p_{err}^{(10)}$ in the narrow gap in Figure 5.5 is due to smoothing in the least squares finite element solver; the expansion solution is more accurate than the finite element solution in this region of the domain. The error patterns that emerge in all these cases are rather interesting, indicating that the spaces \mathcal{H}_{2k} in Theorem 3.3 (the structure theorem) can be quite complicated even for simple curves $h(x)$.

Although our estimates for the error in pressure include an additional factor of $h_0^{-3/2}(r_k + r_k^{-1})^2$, all our numerical experiments (including complicated geometries in which the inf-sup constant β^{-1} does exhibit $h_0^{-3/2}$ behavior) indicate that $\|p_{err}^{(2k)}\|_0$ is comparable to $\|\omega_{err}^{(2k)}\|_0$. In fact, for large k , pressure seems to be the most accurately computed variable; see Figures 5.1 and 5.4. We do not know how to explain this as the pressure is determined by solving (4.92), which involves inverting the operator $\nabla : L_{\#}^2(\Omega) \rightarrow H^{-1}(\Omega)^2$. For some reason, in lubrication-type problems, the right-hand side \mathbf{f}_k belongs to a subspace of $H^{-1}(\Omega)^2$ that is not amplified by $(\nabla)^{-1}$ when solving $\nabla p_{err}^{(2k)} = \mathbf{f}_k$.

The following table shows the minimum ratio of the a priori error estimate to the actual error $\|\mathbf{u}_{err}^{(2k)}\|_{1,\varepsilon}$ for the data points in Figures 5.1 and 5.4 that were used to compute the slopes of the best-fit lines:

$k =$	0	1	2	5	10
(min ratio, $a = 1/5$) $^{1/(2k+2)}$	11.1	7.0	5.0	2.8	2.3
(min ratio, $a = 1/100$) $^{1/(2k+2)}$	34.0	8.6	5.5	3.0	—

For example, in the 10 calculations (with ε ranging from $0.04 \leq \varepsilon \leq 0.099$) that were used to determine the slope of the $2k = 4$ line in Figure 5.1, the ratios of the a priori errors to the exact errors ranged between 1.608×10^4 and 1.617×10^4 , so we recorded $\sqrt[6]{1.608 \times 10^4} \approx 5.0$. This table gives information on how far the values ρ_k in Table 4.4 are from their optimal values. For example, if we increased ρ_5 by more than a factor of 2.8 while holding θ_5 fixed, the estimate (4.110) would fail to hold for this geometry. Since r_k^{-1} in (4.61) is used as a convenient upper bound on all the integrals $|E_{m,j}^{(2\ell)}|^{1/2\ell}$ and $|\tilde{E}_{m,j}^{(2\ell)}|^{1/4\ell}$ that arise in the definition of $Q^{(2k)}$ and also in the bounds for $\|\psi_{xx}^{(2k)}\|_0$ and $\|h^2\psi_{xxxx}^{(2k-2)}\|_0$, it is remarkable that the values of ρ_k we computed are within a factor of 3 of optimal for $k = 5$, $k = 10$, and perhaps all $k \geq 5$.

6. Discussion. Although we are able to estimate the effective radius of convergence $\rho_k r_k$ quite closely, our estimates of $\|\psi_{err}^{(2k)}\|_{2,\varepsilon}$, $\|\omega_{err}^{(2k)}\|_0$, etc., are likely to be several orders of magnitude too large. One shouldn't expect an a priori bound that holds for all geometries alike to provide an exceptionally sharp bound for any specific geometry. Instead, our analysis provides a clear picture of the features of $h(x)$ that cause the effective radii of curvature $r_k \rho_k$ to become small, namely, large values of $h^{k-1} \partial_x^k h$. No previous study has ever described how the constant hidden in the $O(\varepsilon^{2k+2})$ depends on h ; instead, h has always been fixed at the outset and only the limit as $\varepsilon \rightarrow 0$ has been considered.

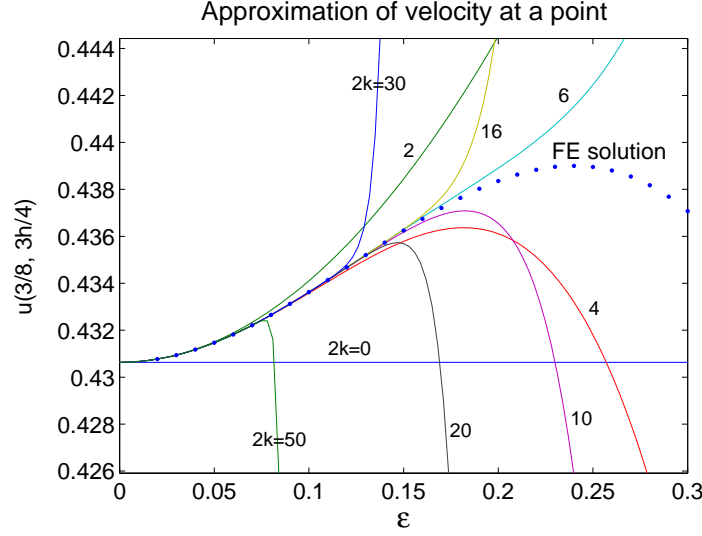


FIG. 6.1. Comparison of $u_{approx}^{(2k)}$ (solid lines) to u_{exact} (dots) at the point $(x, y) = (\frac{3}{8}, \frac{3}{4}h)$ for $2k = 0, 2, 4, 6, 10, 16, 20, 30, 50$. Here $h(x) = \frac{3}{5} + \frac{2}{5}\sin(2\pi x)$, $V_0 = -0.5$, and $V_1 = 1$. This function $h(x)$ is real analytic and periodic, yet the expansion solution appears to be an asymptotic series rather than a convergent series.

Another feature of this analysis is that it separates the constants into two types: those that are (1) given in the problem statement or easily computable from h ; or (2) difficult to compute but universal (independent of h). We listed the first several constants in the latter category (ρ_k and θ_k) in Table 4.4. It is interesting that ρ_k actually increases until $2k = 10$ and doesn't get as bad as ρ_0 again until $2k = 26$. However, at that point it seems to be decreasing steadily like $1/k$, indicating that the effective radius of curvature in our a priori error bound will shrink to zero as $k \rightarrow \infty$. The reason for this is that the recurrences (A.3) and (A.4) relating the matrices $A_i^{(2k)}$ and $B^{(2k)}$ to their lower order counterparts cause the norms of these matrices to grow like $k!$. Thus, although ρ_k involves k th roots of these constants, these k th roots still grow linearly in k . On the other hand, if h is real analytic as well as periodic, a standard contour integral argument shows that there is an $r > 0$ such that $\|\partial_x^k h\|_\infty \leq k! r^{-k}$ for all $k \geq 0$; thus, the constants r_k will remain bounded away from zero. For example, if $h(x)$ is of the form (5.1), one may show that if $a \in (0, 2/3]$, then the largest value of $\|\frac{1}{\ell!} h^{\ell-1} \partial_x^\ell h\|_\infty^{1/\ell}$ occurs when $\ell = 2$, so all the r_k are equal to $r_0 = (\pi\sqrt{1-a})^{-1}$. It is conceivable that when h is real analytic, the norms of the functions $\psi^{(2k)}$ grow slowly enough that the stream function expansion converges in spite of the fact that the matrices $A_i^{(2k)}$ and $B^{(2k)}$ in their representation (3.22) blow up like $k!$. This would simply mean that we chose a bad basis in terms of which to represent ψ . We used orthogonal polynomials in (4.77) to improve this basis, but there may be other improvements. Figure 6.1 shows that this is not the case. Even when h varies sinusoidally, the expansion solution appears to be an asymptotic series rather than a convergent series: all the variables, including the flux terms $Q^{(2k)}$, appear to grow like $k!$ as k becomes large.

Nevertheless, the expansion solutions can be extremely accurate (almost exact) as long as they are used for a geometry that falls within the effective radius of convergence

of the truncated series. It is hoped that the estimates in this paper will help to identify these cases and provide practical a priori (as well as a posteriori) error estimates for many interesting problems.

Appendix A. Implementation. We have developed two methods for computing the higher order corrections described in section 3.2 using a computer. In the first, we use Mathematica to evaluate the derivatives and antiderivatives in recursion (3.6) and Algorithm 3.1 symbolically. With this approach, the main challenge occurs at the step where $Q^{(2k_0)}$ is defined as a definite integral. We do this through pattern matching and symbol replacement. At the stage where the definite integral is to be evaluated, we replace all instances of $\partial_x^j h$ in the integrand by $j! t_j / h^{j-1}$. Each term in the result (call it R) will contain a factor of h^{-3} or h^{-2} , with no other dependence on h . For each $k = k_0, \dots, 0$ and $j = 1, \dots, d_{2k}$, we find the terms in R that contain $\varphi_j^{(2k)}$ (left in the form $t_1^{i_1} \dots t_{2k}^{i_{2k}}$ described in Algorithm 3.2) as a factor. These terms are removed from R while their symbolic integrals (with $\varphi_j^{(2k)} / h^m$ replaced by $I_m E_{m,j}^{(2k)}$) are divided by $2I_3$ and added to the desired flux $Q^{(2k_0)}$. By running through the $\varphi_j^{(2k)}$ in decreasing k order, we convert higher order products (e.g., $t_1^2 t_2 / h^3$) into symbols (e.g., $I_3 E_{3,2}^{(4)}$) before one of their lower order factors can be converted incorrectly (e.g., into $I_3 E_{3,1}^{(2)} t_2$). This approach is effective through 6th or 8th order but becomes rather slow as the complexity of the expansion increases.

The second approach is much faster and can be implemented in any modern programming language. We have written a version in C^{++} and a version in Mathematica. Instead of representing the basis functions $\varphi_j^{(k)}$ for \mathcal{H}_k using a computer algebra system, we represent them as $(k+1)$ -tuples of integers. For example, the functions 1 , h_x , $\frac{1}{6}h^2 h_x h_{xxx}$, and $\frac{1}{48}h^4 h_x h_{xx} h_{xxx}$ in \mathcal{H}_0 , \mathcal{H}_1 , \mathcal{H}_6 , and \mathcal{H}_7 are represented by (0) , $(0, 1)$, $(2, 3, 0, 1, 0, 0, 0)$, and $(4, 1, 1, 0, 1, 0, 0, 0)$. A tuple (i_0, \dots, i_k) represents a basis function for \mathcal{H}_k iff

$$(A.1) \quad i_1 + 2i_2 + \dots + ki_k = k, \quad i_0 = i_2 + 2i_3 + \dots + (k-1)i_k.$$

We begin by constructing the basis sets Φ_k for $0 \leq k \leq 2k_0$ and storing them as $(k+1) \times d_k$ integer matrices with columns corresponding to the $\varphi_j^{(k)}$. This is done using Algorithm 3.2, which returns the columns sorted lexicographically from the last slot to the first slot (e.g., $(3, 0, 3, 0)^T < (2, 3, 0, 1)^T < (3, 1, 1, 1)^T$). Sorted columns allow us to find the column index corresponding to a given tuple in $\log_2 d_k$ time.

Next, for $0 \leq k \leq 2k_0 - 1$, we compute the operators $h\partial_x$ and $h_x \cdot$ from \mathcal{H}_k to \mathcal{H}_{k+1} and store them as sparse integer matrices of dimension $d_{k+1} \times d_k$. If column J of Φ_k contains the tuple (i_0, \dots, i_k) , we define $i_{k+1} = 0$ and compute

$$(A.2) \quad \begin{aligned} h_x \cdot : (i_0, \dots, i_k) &\mapsto (i_0, i_1 + 1, i_2, \dots, i_{k+1}), \\ h\partial_x : (i_0, \dots, i_k) &\mapsto \sum_{\{r : i_r \neq 0\}} i_r(r+1)(i_0 + 1, \dots, i_r - 1, i_{r+1} + 1, \dots, i_{k+1}), \end{aligned}$$

where the omitted indices are unmodified and the $+1$ and -1 cancel in the first slot when $r = 0$ in the sum. The factor of $(r+1)$ is due to the factorials in the definition of the $\varphi_j^{(k)}$. The column index l of each $(k+2)$ -tuple in the result is found in Φ_{k+1} , and the corresponding coefficient $(1 \text{ or } i_r(r+1))$ is added to the l th row and J th column of the sparse matrix representing $h\partial_x$ or $h_x \cdot$. The entries of these sparse matrices are positive, and the column sums (i.e., 1-norms) are all equal to 1 for $h_x \cdot$ and to $i_0 + 2i_1 + \dots + (k+1)i_k = 2k$ for $h\partial_x$ (by (A.1)).

Once the operators $h\partial_x$ and $h_x \cdot$ are known, we use them to recursively compute the matrices $A^{(2k)} = V_0 A_0^{(2k)} + V_1 A_1^{(2k)}$ and $B^{(2k)}$ in (3.22). We start by setting $A_0^{(0)} = (0, 1, -2, 1)^T$, $A_1^{(0)} = (0, 0, -1, 1)^T$, and $B^{(0)} = (0, 0, 3, -2)^T$ as in Example 3.5. For $1 \leq k \leq k_0$, we mimic the proof of Theorem 3.3 to build up $A^{(2k)}$ and $B^{(2k)}$ row by row. For $4 \leq n \leq 2k + 3$ and $i = 0, 1$, we use sparse matrix–vector multiplication to define the rows

$$(A.3) \quad \begin{aligned} A_i^{(2k)}(n, :) &= \left(\frac{-2[h\partial_x - (n-2)h_x][h\partial_x - (n-3)h_x] [A_i^{(2k-2)}(n, :)^T]}{n(n-1)} \right)^T, \\ B^{(2k)}(n, :) &= \left(\frac{-2[h\partial_x - (n-1)h_x][h\partial_x - (n-2)h_x] [B^{(2k-2)}(n, :)^T]}{n(n-1)} \right)^T. \end{aligned}$$

If $k \geq 2$, then for $6 \leq n \leq 2k + 3$, we add the following vectors to $A_i^{(2k)}(n, :)$ and $B^{(2k)}(n, :)$, respectively:

$$(A.4) \quad \begin{aligned} &\left(\frac{-[h\partial_x - (n-2)h_x][h\partial_x - (n-3)h_x][h\partial_x - (n-4)h_x][h\partial_x - (n-5)h_x] [A_i^{(2k-4)}(n, :)^T]}{n(n-1)(n-2)(n-3)} \right)^T, \\ &\left(\frac{-[h\partial_x - (n-1)h_x][h\partial_x - (n-2)h_x][h\partial_x - (n-3)h_x][h\partial_x - (n-4)h_x] [B^{(2k-4)}(n, :)^T]}{n(n-1)(n-2)(n-3)} \right)^T. \end{aligned}$$

Next we zero out rows 0 and 1 of $A_0^{(2k)}$, $A_1^{(2k)}$, $B^{(2k)}$, and set

$$(A.5) \quad \begin{aligned} A_i^{(2k)}(2, :) &= \sum_{n=4}^{2k+3} (n-3) A_i^{(2k)}(n, :), & A_i^{(2k)}(3, :) &= \sum_{n=4}^{2k+3} (2-n) A_i^{(2k)}(n, :), \\ B^{(2k)}(2, :) &= \sum_{n=4}^{2k+3} (n-3) B^{(2k)}(n, :), & B^{(2k)}(3, :) &= \sum_{n=4}^{2k+3} (2-n) B^{(2k)}(n, :). \end{aligned}$$

Finally, we subtract $\binom{-1/2}{k}$ from $A_1^{(2k)}(2, 1)$ and add it to $A_1^{(2k)}(3, 1)$ to account for the boundary data, where we recall that the rows and columns are indexed starting at 0 and 1, respectively. Using this approach, our C^{++} code can compute these matrices through order $2k = 50$ using floating point arithmetic in a few seconds, while our Mathematica code can compute through order $2k = 30$ in exact rational arithmetic in about an hour. This allows us to explore the properties of the stream function expansion and test our error estimates to quite a high order.

REFERENCES

- [1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1964.
- [2] A. ASSEMIEN, G. BAYADA, AND M. CHAMBAT, *Inertial effects in the asymptotic behavior of a thin film flow*, *Asympt. Anal.*, 9 (1994), pp. 177–208.
- [3] G. BAYADA AND M. CHAMBAT, *The transition between the Stokes equations and the Reynolds equation: A mathematical proof*, *Appl. Math. Optim.*, 14 (1986), pp. 73–93.
- [4] G. BAYADA AND M. CHAMBAT, *Modélisation de la jonction d'un écoulement tridimensionnel et d'un film mince bidimensionnel*, *C. R. Acad. Sci. Paris, Ser. I*, 309 (1989), pp. 81–84.
- [5] D. BRAESS, *Finite Elements—Theory, Fast Solvers, and Applications in Solid Mechanics*, Cambridge University Press, Cambridge, UK, 1997.

- [6] Z. CAI, T. A. MANTEUFFEL, AND S. F. MCCORMICK, *First-order system least squares for the Stokes equations, with application to linear elasticity*, SIAM J. Numer. Anal., 34 (1997), pp. 1727–1741.
- [7] E. V. CHIZHONKOV AND M. A. OLSHANSKII, *On the domain geometry dependence of the LBB condition*, M2AN Math. Model. Numer. Anal., 34 (2000), pp. 935–951.
- [8] G. CIMATTI, *How the Reynolds equation is related to the Stokes equations*, Appl. Math. Optim., 10 (1983), pp. 267–274.
- [9] I. CIUPERCA, I. HAFIDI, AND M. JAI, *Singular perturbation problem for the incompressible Reynolds equation*, Electron. J. Differential Equations, 2006 (2006), pp. 1–19.
- [10] M. DOBROWOLSKI, *On the LBB constant on stretched domains*, Math. Nachr., 254–255 (2003), pp. 64–67.
- [11] A. DUVNJAK AND E. MARUŠIĆ-PALOKA, *Derivation of the Reynolds equation for lubrication of a rotating shaft*, Arch. Math., 36 (2000), pp. 239–253.
- [12] H. G. ELROD, *A derivation of the basic equations for hydrodynamic lubrication with a fluid having constant properties*, Quart. Appl. Math., XVII (1960), pp. 349–359.
- [13] G. P. GALDI, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. 1: Linearized Steady Problems*, Springer-Verlag, New York, 1994.
- [14] V. GIRAULT AND P.-A. RAVIART, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, 1986.
- [15] J. KEVORKIAN AND J. D. COLE, *Multiple Scale and Singular Perturbation Methods*, Springer-Verlag, New York, 1996.
- [16] W. E. LANGLOIS, *Slow Viscous Flow*, Macmillan, New York, 1964.
- [17] I. MOISE, R. TEMAM, AND M. ZIANE, *Asymptotic analysis of the Navier-Stokes equations in thin domains*, Topol. Methods Nonlinear Anal., 10 (1997), pp. 249–282.
- [18] S. A. NAZAROV, *Asymptotic solution of the Navier-Stokes problem on the flow of a thin layer of fluid*, Siberian Math. J., 31 (1990), pp. 296–307.
- [19] S. A. NAZAROV, *Asymptotics of the Stokes system solutions at a surfaces contact point*, C. R. Acad. Sci. Paris, Sér. I, 312 (1991), pp. 207–211.
- [20] C. POZRIKIDIS, *Introduction to Theoretical and Computational Fluid Dynamics*, Oxford University Press, New York, 1997.
- [21] G. RAUGEL AND G. R. SELL, *Navier-Stokes equations on thin 3D domains. I: Global attractors and global regularity of solutions*, J. Amer. Math. Soc., 6 (1993), pp. 503–568.
- [22] O. REYNOLDS, *On the theory of lubrication and its applications to Mr. Beauchamp Tower's experiments, including an experimental determination of the viscosity of olive oil*, Philos. Trans. R. Soc. Lond., 177 (1886), pp. 157–234.
- [23] G. STOYAN, *Iterative Stokes solvers in the harmonic Velt subspace*, Computing, 67 (2001), pp. 13–33.
- [24] J. WILKENING, *Inf-sup estimates for the Stokes problem in a periodic channel*, arXiv:0706.4082.
- [25] J. WILKENING AND A. E. HOSOI, *Shape optimization of a sheet swimming over a thin liquid layer*, J. Fluid Mech., 601 (2008), pp. 25–61.